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Point and Confidence Interval Estimation of the Parameter and Survival Function for Lindley Distribution Under Censored and Uncensored Data

Lindley Dağılımında Sansürlü ve Sansürsüz Verilerde Parametre ve Sağ Kalım Fonksiyonları için Nokta ve Aralık Tahmini

Kamil ALAKUŞ[a],
Necati Alp ERİLLİ[b]

[a]Department of Statistics,
Ondokuz Mayis University
Faculty of Arts and Sciences,
Samsun, TURKEY
[b]Department of Econometrics,
Sivas Cumhuriyet University
Faculty of Economic and Administrative,
Sivas, TURKEY

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Correspondence:
Necati Alp ERİLLİ
Sivas Cumhuriyet University,
Department of Econometrics,
Faculty of Economic and Administrative
Sivas Cumhuriyet University, Sivas,
TURKEY/TÜRKİYE
aerilli@cumhuriyet.edu.tr

ABSTRACT Objective: Survival analysis is a statistical method used to compare the life expectancy of patients in health studies and to investigate the effectiveness of the treatments. The Lindley distribution is a widely used distribution in survival analysis in recent years and is used to describe the life of a process or device. The Lindley distribution is a two-parameter continuous distribution that is widely used in a wide range of fields including biology, engineering and medicine. **Material and Method:** In this study, point and interval estimates of the Lindley distribution were examined for censored and uncensored data. Parameter inferences and confidence intervals are shown. In the application section, real-time data (uncensored) and a simulation data (uncensored and censored) are compared with other distributions of the exponential distribution family. **Results:** According to the results obtained from the analysis, it was observed that Lindley distribution gave better results in special data structures compared to other distributions of exponential family. Lindley distribution; For uncensored real-life data and censored and uncensored simulation data, it has a lower model selection criterion. **Conclusion:** The exact distribution of the data to be used in survival analysis is one of the prerequisites for the success of the analysis. The Lindley distribution, with its increasing popularity in recent years, is a distribution that gives successful results in survival analysis

Keywords: Lindley Distribution; Survival Analysis; Censored Data; Point Estimation

ÖZET Amaç: Sağ Kalım analizi, sağlık çalışmalarında hastaların beklenen yaşam sürelerinin karşılaştırılmasında, uygulanacak tedavilerin etkinlikleinin araştırılmasında kullanılan istatistiksel yöntemlerdir. Lindley dağılımı, son yıllarda sağ kalım analizlerinde sık kullanılan bir dağılımdır ve bir işlem ya da cihazın ömrünü tanımlamada kullanılır. Biyoloji, mühendislik ve tıp da dahil olmak üzere çok çeşitli alanlarda uygulama alanı bulan Lindley dağılımı, iki parametreli sürekli bir dağılımdır. **Gereç ve Yöntemler:** Bu çalışmada, sansürlü ve sansürsüz veriler için Lindley dağılımının nokta ve aralık tahminleri incelenmiştir. Parametre çıkarımları ve güven aralıklarının bulunması gösterilmiştir. Uygulama bölümünde ise gerçek zamanlı veriler (sansürsüz) ve bir simülasyon verisi (sansürsüz ve sansürlü), üstel dağılım ailesinin diğer dağılımları ile karşılaştırılmıştır. **Bulgular:** Analizlerden elde edilen sonuçlara göre, Lindley dağılımının özel veri yapılarında üstel ailesinin diğer dağılımları ile karşılaştırıldığında daha iyi sonuçlar verdiği görülmüştür. Lindley dağılımı; Sansürsüz gerçek yaşam verisi ile sansürlü ve sansürsüz simülasyon verilerinde, daha düşük model seçme kriterine sahiptir. **Sonuç:** Sağkalım analizlerinde kullanılacak verilerin dağılımlarının tam olarak belirlenmesi, analizin daha başarılı olması için gereken ön şartlarından biridir. Lindley dağılımı da son yıllarda artan popülerliği ile sağ kalım analizlerinde başarılı sonuçlar veren bir dağılımdır.

Anahtar Kelimeler: Lindley Dağılımı; Sağ Kalım Analizi; Sansürlü Veri; Nokta Tahmini

Survival analysis; is used to analyse data over time until any event we are interested in. In Survival Analysis, the problem can be death, epidemic, machine breakdown, divorce, etc. Survival time can be measured in hours, days, or years, etc. For example, if the event of research is paralysis, the survival time can be the measured in years until a person develops a paralysis.[1] Survival period refers to the life span of individuals until death or to the last known date. Survival analysis is also expressed as failure analysis, life analysis or event time analysis and it shows the time from a specific starting point to the moment when an event takes place. In survival analysis, there are different approaches to the solution of the issue at hand. The most important and the most used of these approaches is to make predictions about the analysis made by using various survival distributions and to prepare hypothesis tests. In survival analyses, the most used distributions are exponential family distributions. Although Weibull, Gamma, Log-Normal and exponentiated exponential are the most frequently used ones, using special distributions introduced for special data structures give better results.[2,3,4]

However, these four distributions of the exponential family (gamma, lognormal, Weibull and exponential) have some disadvantages. The important disadvantage is that none of them exhibit bathtub shapes for their hazard rate functions. The four distributions exhibit only monotonically increasing, monotonically decreasing or constant hazard rates. This is an important problem just because most real-life systems exhibit bathtub shapes for their hazard rate functions. Second problem is more of the four distributions can exhibit constant hazard rates. This is a very unlikely and unrealistic feature because real life systems with constant hazard rates are almost non-existent.[5]

One of these distributions is Lindley distribution introduced by Lindley.[6] Probability intensity function of the function is as given in Equation.1:

$$f(t;\beta) = \frac{1}{\beta(\beta+1)}\,(t+1)e^{-\frac{t}{\beta}},\ t>0 \tag{1}$$

There are many studies with Lindley distribution in the literature. These are special cases of the Lindley distribution (Power Lindley, Quasi Lindley, Weibull-Lindley, Extended Lindley, Poisson Lindley, Discrete Lindley, Transmuted Lindley, Generalized Lindley, etc.) and most uncensored data were studied. In this study, inferences of parameter estimates for Lindley distribution were found separately for both censored and uncensored data. The aim of this paper is to introduce an extension of the Lindley distribution which offers a more flexible distribution for modeling life time data, namely in reliability, in terms of its failure rate shapes with cencored and uncensored data.

In this study it is introduced the point and confidence interval estimation of the parameter and survival function of Lindley distribution with single parameter in censored and non-censored data. In section 2, distribution and functional properties of Lindley distribution is discussed. General incidentally censored sample data is also discussed in section 3. Section 4 deals with the parametric maximum likelihood estimation and survival function properties, observed Fisher information, interval estimation of the parameter based on the normal approach of the distribution. In the application section of the study, power of distribution is tested with a real-life and a censorship data. Results are compared with other exponential-family distribution results.

MATERIAL AND METHODS

THE LINDLEY DISTRIBUTION

The Lindley distribution was originally proposed and developed by Lindley to analyze failure time data that has been modelled stress-resistance reliability.[6] The Lindley distribution is a mono-parameter unidirectional distribution that is used to analyze the life span of a process or device and the survival time of a living thing. It can be used in a variety of fields including biology, engineering and medicine. Ghitany et al.[7] stated that this distribution is particularly useful in modeling survival studies. The shape parameter exhibits a distribution image with single-modal or monotonically decreasing (ie, continuously decreasing) probability density function, since β is a positive real number.

In survival analysis studies, one sample of *N* observer sample is tested, and when all fail, the experiment is terminated. This process can take a long time if the distribution of the survival time of the units has a more curved and thicker tail. Moreover, if items such as medical equipment are expensive, collecting all the sample information is costly. There are many situations in which experimental units are removed from the test before they are removed or completely removed from the test. For example, in a clinical trial, the individual may stop working, the source of the test may disappear, the work may be prematurely terminated due to lack of resources, or the test environment may be damaged by other factors. The test units may be accidentally broken. In other scenarios, the test may need to be terminated to release the test for other purposes. When the situations outlined above are encountered, censored data is used in the survival analysis to save the time and costs of the test units. Units in a test can be removed unintentionally or in advance. The data obtained from such experiments or researches are called censored samples. The survival times are summed to estimate the parameters and reliability functions. In most studies, detailed information about the sample is not available. This can create problems in the estimation process. In such cases, the failure information of the censored observations can be used up to a predetermined time.

There are many censoring patterns encountered in survival analysis. Type I and type II censorship are the most popular. In addition, there are all kinds of censorship in the general censorship type, especially encountered in survival studies. However, these plans do not allow the units to be removed from the system before the trial is terminated. For this reason, we dealt with a general type of censorship. With the information from an old records for any data is possible to obtain indicators to estimate, calculate and understand the behavior of the equipments failures. Therefore, it will be possible to determine the appropriate analysis methods for any equipment and components using appropriate methods.

In this type of studies, if the failure time in the system is known exactly, the data is defined as complete data. In many cases, there may be data with uncertain conditions. In these cases, the exact moment of the error cannot be known. The data containing such uncertainty as to when the event occurred are defined as incomplete or partial. Missing data is defined as censored data, but in some studies it is called cut data. There is no rule that missing data will always give all information about the downtime of the units being examined. In some analyzes, we can only retrieve some of the information. However, this little information should not be considered a failure. In the absence of such data, it is not easy to make good estimation parameters and therefore to make an appropriate analysis.

One of the most common types of censored data, which may arise in real cases, is Type-1 right censored data.[8] In type-1 right-censored data, all units should be observed until the date of completion of the study or event. For this censorship application, the time of each unit is kept under observation when all are fixed. For these, the number of failing units in the system can be random. Let *T* is a random variable representing the failure time and *C* is another random variable independent of *T* and corresponds to the end of the observation time. Thus we can say that the time to failure is right censored when one does not know its real value, only that its value is bigger than *C*, with regard to item *i* ($i = 1,2, \ldots, n$). Therefore, *T* and *C* can be described as; $t_i = \min(T_i, C_i)$ and $\delta_i = \begin{cases} 1 \text{ if } T_i \leq C_i \\ 0 \text{ if } T_i > C_i \end{cases}$. Here; The δ_i variable (censorship indicator) indicates whether T_i is censored or not. The obtained data can be represented by the pair (t_i, δ_i) i.e. t_i the failure time or censored time and δ_i the variable that indicates whether it concerns a failure or censorship is written as, $\delta_i = \begin{cases} 1 \text{ for uncensored data} \\ 0 \text{ for censored data} \end{cases}$. In the right censored data the failure time of the units with censored data it is just known to be greater than the operating time of the conclusion of the registration information. Right-censored data is classified as Type-1 and Type-2 censorship if a previously recorded error occurs and the information record is interrupted in any previous period of study.[8]

Functions and Some Distributional Characteristics

Distribution function which is obtained with the help of Probability Intensity function of Lindley distribution given with Equation (1) is obtained with Equation 2, survival function is obtained with Equation 3 and hazard function is obtained with Equation 4.

$$F(t;\beta) = 1-\left[\frac{(\beta+t+1)}{(\beta+1)} \right] e^{-\frac{t}{\beta}}, t>0 \quad (2)$$

$$S(t;\beta) = 1-F(t;\beta) = \left[\frac{(\beta+t+1)}{(\beta+1)} \right] e^{-\frac{t}{\beta}}, t>0 \quad (3)$$

$$h(t;\beta) = \frac{f(t;\beta)}{S(t;\beta)} = \left[\frac{(t+1)}{\beta\,(\beta+t+1)} \right] e^{-\frac{t}{\beta}}, t>0 \quad (4)$$

The average and variance of the distribution, k. moment of which according to origin is given in Equation 5, are given in Equation 6 and Equation 7, respectively.

$$\mu_k = E(T^k) = \frac{k\beta^k\left[(k+1)\,\beta+1\right]}{(\beta+1)}, k = 1,2,... \quad (5)$$

$$\mu = \mu_k = E(T) = \frac{\beta\left[2\beta+1\right]}{(\beta+1)} \quad (6)$$

$$\sigma^2 = \mu_2 - (\mu_1)^2 = E(T^2) - \left[E(T)\right]^2 = \frac{\beta^2\left[2\beta^2+4\beta+1\right]}{(\beta+1)^2} \quad (7)$$

Uncensored Parameter Estimation of Lindley Distribution

In this section, maximum likelihood method will be used for the point estimation of β. If T random variable has one parameter Lindley distribution given with (1), probability function is given with:

$$L(t;\beta) = \prod_{i=1}^{n} f(t_i;\beta) = \beta^{-n}\,(\beta+1)^{-n}\, e^{-\frac{\sum_{i=1}^{n} t_i}{\beta}} \prod\,(t_i+1) \quad (8)$$

Log-probability function is given with:

$$\ln L(t;\beta) = \sum_{i=1}^{n} \ln\left[f(t_i;\beta)\right] = -n\ln(\beta) - n\ln(\beta+1) + \sum_{i=1}^{n}\ln(t_i+1) - \frac{\sum_{i=1}^{n} t_i}{\beta} \quad (9)$$

In Equation (9), if derivative is calculated according to β and assessed at zero, the following homogeneous equation is formed:

$$\frac{\partial \ln L(t;\beta)}{\partial\beta} = -\frac{n}{\beta} - \frac{n}{(\beta+1)} + \frac{n\bar{t}}{\beta^2} = 0 \quad (10)$$

If this equation is organized,

$$2\beta^2 + (1-\bar{t})\beta - \bar{t} = 0 \quad (11)$$

it becomes second degree polynomial of β.Since β is positive, maximum likelihood estimator will be found as

$$\hat{\beta} = \frac{(\bar{t}-1) + \sqrt{(1-\bar{t})^2 + 8\bar{t}}}{4} \quad (12)$$

β's variance and standard error is differentiated a second time in the function given with (9) according to β and Hessian matrix, $H = \frac{\partial^2 \ln L(t;\beta)}{\partial\beta^2} = \frac{n}{\beta^2} - \frac{n}{(\beta+1)^2} + \frac{2n\bar{t}}{\beta^3}$ is found. Here, information matrix $I(\beta) = -H(\beta) = -\frac{n}{\beta^2} - \frac{n}{(\beta+1)^2} + \frac{2n\bar{t}}{\beta^3}$ is obtained. The reverse of this information matrix is the variance of β. That is, $V(\hat{\beta}) = I(\beta)^{-1} = \frac{\beta^3(\beta+1)^2}{n\left\{(\beta+1)\left[2(\beta+1)\bar{t} - \beta\right] - /\beta^3\right\}}$ is found. Standard error of β is the positive square root of this quantity.

For Interval estimation of Lindley distribution, it used confidence interval based on probability rate test. Let the value of log probability function given with Function (9) in $\hat{\beta}$ be $LL(\hat{\beta})$. 100 (1-α)% confidence and 1 d.f. table value of chi-square distribution is χ^2_{Table} and half of this value is $LL(\hat{\beta}) - LL(\hat{\beta}_{Lower}) = 0{,}5 \times \chi^2_{Table}$ and $LL(\hat{\beta}) - LL(\hat{\beta}_{Upper}) = 0{,}5 \times \chi^2_{Table}$ which confirms half of this value $0{,}5 \times X^2_{Table}$. For example, this value is 3.841 with 95% and 1 d.f. and half of it is 1.9205. The estimation of lower limit of survival function is given with $\hat{S}_{LD}(t) = \frac{(\hat{\beta}_{Lower} + t_i + 1)}{(\hat{\beta}_{Lower} + 1)} exp\left(-\frac{t_i}{\hat{\beta}_{Lower}}\right)$ while the estimation of upper limit of survival function is given with $\hat{S}_{UL}(t) = \frac{(\hat{\beta}_{Upper} + t_i + 1)}{(\hat{\beta}_{Upper} + 1)} exp\left(-\frac{t_i}{\hat{\beta}_{Upper}}\right)$.

Let β's asymptotic variance estimation be $V(\hat{\beta})$ and its standard error be $sh(\hat{\beta})$. β's 100 (1-α)% lower and upper confidence limits are given with $(\hat{\beta}_{Lower}) = \hat{\beta} - z_{\alpha/2} h(\hat{\beta})$ and $(\hat{\beta}_{Upper}) = \hat{\beta} + z_{\alpha/2} h(\hat{\beta})$, respectively. 100 (1-α)% confidence limits of survival function suitable for these intervals are written as lower limit: $\hat{S}_{LD}(t) = \frac{(\hat{\beta}_{Lower} + t_i + 1)}{(\hat{\beta}_{Lower} + 1)} exp\left(-\frac{t_i}{\hat{\beta}_{Lower}}\right)$ and upper limit: $\hat{S}_{UL}(t) = \frac{(\hat{\beta}_{Upper} + t_i + 1)}{(\hat{\beta}_{Upper} + 1)} exp\left(-\frac{t_i}{\hat{\beta}_{Upper}}\right)$.

Censored Parameter Estimation of Lindley Distribution

For censored survival times, let d from a sample with n sample size be the individual's time of death $(t_1,...,t_d)$ and $(n\text{-}d)$ be the individual's censored survival time $(c_1,...,c_{n-d})$. Now, let $x_i = Smallest\{t_i, c_i\}$ random variable be defined. Thus, total probability function can be written as given in Equation.13:

$$L(x_i;\theta) = \prod_{i=1}^{n} \left[f(x_i;\theta) \right]^{w_i} \left[S(x_i;\theta) \right]^{1-w_i} \tag{13}$$

In this function, δ_i is the censor marker variable and is defined as

$$w_i = \begin{cases} 0; \textit{ observation is censored} \\ 1; \textit{ observation is uncensored} \end{cases} \tag{14}$$

That is, it is in the form of dummy variable. In Function (13), the first factor is taken from the survival time of individuals who die, while the second factor is taken from the censored survival time. The log likelihood function suitable for Function (13) is as follows:

$$\text{In } L(x_i;\theta) = \sum_{i=1}^{n} \delta_i \text{In } f(x_i;\theta) + \sum_{i=1}^{n} (1-\delta_i) \text{ In } S(x_i;\theta) \tag{15}$$

In this section, maximum likelihood method will be used for the estimation of β. If t random variable has the one parameter Lindley distribution given with (1), likelihood function given with (13) is given as follows:

$$L(t;\beta) = \beta^{-d} (\beta+1)^{-n} exp \left(\frac{\sum_{i=1}^{n} t_i}{\beta} \right) \prod_{i=1}^{n} (t_i+1)^{w_i} \prod_{i=1}^{n} (\beta+t_i+1)^{1-w_i} \tag{16}$$

Log-likelihood function is given as follows:

$$\text{In } L(t;\beta) = -d \text{ In } (\beta) - n\text{In}(\beta+1) + \sum_{i=1}^{n} w_i \text{In } (t_i+1) - \frac{\sum_{i=1}^{n} t_i}{\beta} + \sum_{i=1}^{n} (1-w_i) \text{ In } (\beta+t_i+1) \tag{17}$$

In Equation (17), if the derivative is taken according to β and assessed at zero,

$$\frac{\partial \text{In} L(t;\beta)}{\partial \beta} = -\frac{d}{\beta} - \frac{n}{(\beta+1)} + \frac{n\bar{t}}{\beta^2} + \sum_{i=1}^{n} (1-w_i) \frac{1}{(\beta+t_i+1)} = 0 \tag{18}$$

the value which meets the equation becomes MLE estimation of β.

In terms of β's variance and standard error, Hessian matrix $H = \frac{\partial^2 \text{In} L(t_i;\beta)}{\partial^2 \beta} = \frac{d}{\beta^2} + \frac{n}{(\beta+1)^2} - \frac{2n\bar{t}}{\beta^3} - \sum_{i=1}^{n} (1-w_i) \frac{1}{(\beta+t_i+1)^2}$ is found by taking derivative according to β for the second time given with (18). Here, information matrix is obtained with $I(\beta) = -H(\beta) = -\frac{d}{\beta^2} - \frac{n}{(\beta+1)^2} + \frac{2n\bar{t}}{\beta^3} + \sum_{i=1}^{n} (1-w_i) \frac{1}{(\beta+t_i+1)^2}$. If $S_1 = \frac{d}{\beta^2}$, $S_2 = \frac{n}{(\beta+1)^2}$,

$S_3 = \frac{2n\bar{t}}{\beta^3}$ and $S_4 = \sum_{i=1}^{n} (1-w_i) \frac{1}{(\beta+t_i+1)^2}$ then, $I(\beta) = S_3 + S_4 - S_1 - S_2$. This information is the variance of β, which is the contrary of information matrix. That is, $V(\hat{\beta}) = I(\beta)^{-1} = \frac{1}{(S_3 + S_4 - S_1 - S_2)}$. β's standard error is the positive square root of this quantity.

Short Literature Review

Recently, a great number of studies have been conducted in literature on Lindley distribution. Shanker and Shukla discussed about zero-truncated two-parameter Poisson-Lindley distribution which includes zero-truncated Poisson-Lindley distribution as a particular case has been obtained by compounding size-biased Poisson distribution with an assumed continuous distribution Shanker and Mishra[10] introduced a two-parameter Quasi Lindley distribution (QLD) which is a particular case of the Lindley distribution. Shanker et al. proposed a three-parameter Lindley distribution, which includes some two-parameter Lindley distributions, two-parameter gamma distribution, and one parameter exponential and Lindley distributions as special cases, for modelling lifetime data.[9-11] Bhati et al. introduced a new class of distributions generated by an integral transform of the probability density function of the Lindley distribution which results ina model that is more flexible in the sense that the derived model spans distributions with increasing failure rate, decreasing failure rate and upside down bathtub shaped hazard rate functions for different choices of parametric values.[12] Shanker and Mishra has introduced a two-parameter Lindley distribution, of which the Lindley distribution is a particular case and also Its moments, failure rate function, mean residual life function and stochastic orderings have been discussed.[13] Altun et al. introduce a new model called the Odd Burr Lindley distribution which extends the Lindley distribution and has increasing, bathtub and upside down shapes for the hazard rate function.[14] Coelho-Barros et al. Studied on classical and Bayesian inference methods for to analyse lifetime data sets in the presence of left censoring considering two generalizations of the Lindley distribution.[15] Mazucheli et al. compared through Monte Carlo simulations the finite sample properties of the estimates of the parameters of the weighted Lindley distribution obtained by four estimation methods: maximum likelihood, method of moments, ordinary least-squares, and weighted least-squares.[16] Ashour and Eltehiwy proposed a more generalization of the Lindley distribution which generalizes generalization of the Lindley distribution introduced by Ghitany et al. and Nadarajah et al. Respectively.[17] Zakerzadeh and Dolati introduced three parameter generalization of the Lindley distribution, thus this includes as special cases the exponential and gamma distributions.[18] Cakmakyapan and Ozel proposed a new class of distributions called the Lindley generator with one extra parameter to generate many continuous distributions.[19] Alizadeh et al. introduced a four-parameter distribution, called odd Burr power Lindley distribution, which extends the Lindley distribution and has increasing, upside-down and bathtub shapes for the hazard rate function.[20] Merovci and Sharma introduced the new continuous distribution, so-called the Beta-Lindley distribution that extends the Lindley distribution.[21] We provide a comprehensive mathematical treatment of this distribution with deriving the moment generating function and the r-th moment thus generalizing some results in the literature. Zamani and Ismail introduced a new mixed negative binomial distribution by mixing the distributions of negative binomial and Lindley, where the reparameterization of $p = \exp(-\lambda)$ is considered.[22]

RESULTS

In this section, two real time (uncensored) and three simulations (uncensored and censored) numerical examples are tested. The results obtained from Lindley distribution were compared with those obtained from the Exponential, Gamma, Log-Normal, Log-logistic, Weibull and Pareto distributions which are

TABLE 1: Bank customers before service waiting time.[27]

Waiting time (In minutes)														
0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3	3.5
3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1
7.1	7.4	7.6	7.7	8.0	8.2	8.6	8.6	8.8	8.8	8.9	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4	12.5	12.9	13.0
13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5					

TABLE 2: Bank queue data probability distribution estimates.

Distribution	Parameter Estimates		LL	AIC	AICc	BIC
Exponential	$\hat{\beta}$=9.8770		-329.0209	660.0418	660.0826	662.6470
Lindley	$\hat{\beta}$=5.359882		-319.0374	640.0748	640.1156	642.6800
Gamma	$\hat{\alpha}$=2.008663	$\hat{\beta}$=4.917201	-317.3001	638.6002	638.7240	643.8105
Weibull	$\hat{\alpha}$=1.458488	$\hat{\beta}$=10.955316	-318.7307	641.4614	641.5851	646.6717
Log-Normal	$\hat{\mu}$=2.011116	$\hat{\sigma}^2$=0.608578	-319.1740	642.3480	642.4717	647.5583
Log-Logistic	$\hat{\alpha}$=2.267123	$\hat{\beta}$=7.815883	-319.4098	642.8196	642.9433	648.0299
Pareto	$\hat{\alpha}$=0.445581	$\hat{\beta}$=0.8	-382.9492	769.8985	770.0222	775.1088

frequently used in survival analysis. For comparison, 4 different model selection criteria are used. These are; Akaike (AIC), Corrected Akaike (AICc), Bayesian (BIC), Log-Likelihood (LL).[23-26]

Real-Time Data Example-I

In real time application, it is used a data set corresponding to waiting times (in minutes) before service of 100 bank customers as discussed by Ghitany et al. The data are given in Table.1.[27]

The results of the 7 methods mentioned above applied to the data given in Table.1 are given in Table.2. When we look at the results in Table 2, the results obtained from Gamma distribution are best according to AIC, AICC and LL criteria. The prediction obtained from the Lindley distribution gave the best result according to the BIC criterion and the second-best result according to the AIC and AICC criteria.

Lindley Distribution parameter estimate and Log-Likelihood criteria for Lindley estimation is given respectively;

$$t = \frac{\sum_{i=1}^{n} t_i}{n} = \frac{987,7}{100} = 9,877; \hat{\beta} = \frac{(\bar{t}-1)+\sqrt{(1-\bar{t})^2+8\,\bar{t}}}{4} = 5,359882$$

$$LL(\hat{\beta} = 5,359882) = -n\text{xIn}\,(\hat{\beta}) - n\text{xIn}\,(\hat{\beta}+1)+\sum_{i=1}^{100}\text{In}\,(t_i+1) - \frac{\sum_{i=1}^{n} t_i}{\hat{\beta}}$$

$$=-100xIn\ (5{,}359882)\ -100xIn\ (5{,}359882+1)+218{,}1342-\frac{987{,}7}{5{,}359882}=-319{,}0374$$

Thus, survival function estimate is given by $\hat{S}(t)=\frac{(5{,}359882+t_i+1)}{(5{,}359882+1)}\exp\left(-\frac{t_i}{5{,}359882}\right)$.

Confidence Interval Based on Probability Rate Test (95%)

The lower and upper limits of β, respectively are $LL(\hat{\beta}_{Lower}=4{,}676795)=-320{,}9574$ and $LL(\hat{\beta}_{UPPER}=6{,}181847)=-320{,}9574$.

The estimate of survival function lower limit is found by $\hat{S}_{LD}(t)=\frac{(4{,}676795+t_i+1)}{(4{,}676795+1)}\exp\left(-\frac{t_i}{4{,}676795}\right)$, while the upper limit estimate is found by $\hat{S}_{UL}(t)=\frac{(6{,}181847+t_i+1)}{(6{,}181847+1)}\exp\left(-\frac{t_i}{6{,}181847}\right)$.

Confidence Interval Based On Normal Distribution (Asymptotic - 95%)

Asymptotic variance estimates of β is found as $V(\hat{\beta})=0{,}14544$, while its standard error is found as $Se(\hat{\beta})=0{,}381365$. 95% lower and upper limits of β are found as $\hat{\beta}_{Lower}$=5,3598-1,96 x (0,3813) =4,612 and $\hat{\beta}_{UPPER}$=5,3598-1,96 x (0,3813) =6,107, respectively. 95% confidence limits of survival function suitable for these intervals are written as:

Lower limit: $\hat{S}_{LD}(t)=\frac{(4{,}612406+t_i+1)}{(4{,}612406+1)}\exp\left(-\frac{t_i}{4{,}612406}\right)$

and higher limit: $\hat{S}_{UL}(t)=\frac{(6{,}107358+t_i+1)}{(6{,}107358+1)}\exp\left(-\frac{t_i}{6{,}107358}\right)$

It is given hazard and survival functions with %95 confidence intervals for Bank Customer Service Waiting Times in Figure.1.

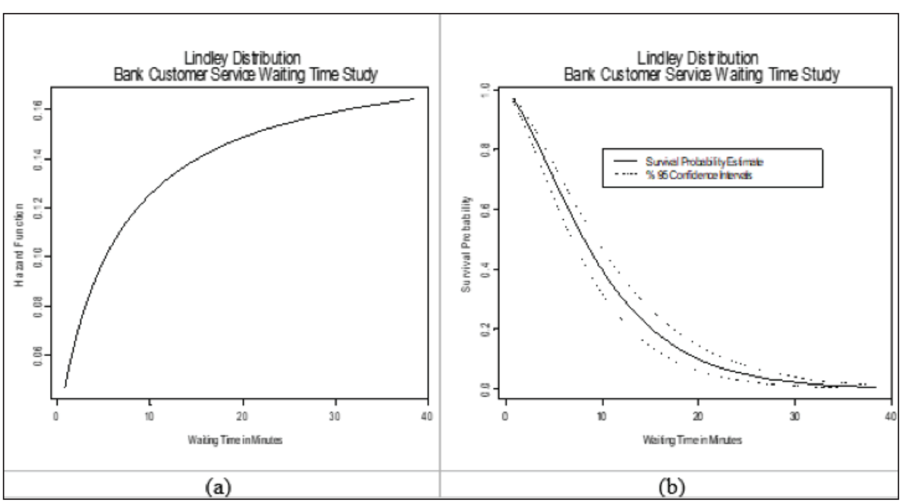

FIGURE 1: For Bank Customer Service Waiting Times: (a) hazard and (b) survival functions with 95% CI's.

Real-Time Data Example-II

The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm.[28] Data is given in Table.3:

The results of the 7 methods mentioned above applied to the data given in Table.3 are given in Table.4. As for the results given Table 4, it is clearly seen that Weibull distribution is the best according to all criterias. Since $\alpha>4$, the data is best suited for normal distribution.

But when we apply Power Lindley to the data (Lindley's special case) we can see that the best log-likelihood result is passed from Weibull to Power Lindley Distribution:

Estimation for Power Lindley: $\hat{\alpha} = 3.868$; $\hat{\beta} = 0.050$; $LL =$ -49.059.

It should be noted here that there is a special case where two-parameter Lindley distribution (power Lindley) gives more successful results in distributions where single parameter Lindley distribution is insufficient. The Lindley distribution does not provide enough flexibility for analysing different types of lifetime data because of having only one parameter. To increase the flexibility for modelling purposes it will be useful to consider further alternatives of this distribution. In Table.5, it is given confidence intervals for Lindley coefficient given in Table.4:

TABLE 3: Tensile Strength, measured in GPa, of 69 carbon fibers.[28]

1.312	1.966	2.224	2.382	2.566	2.770	3.067
1.314	1.997	2.240	2.426	2.57	2.773	3.084
1.479	2.006	2.253	2.434	2.586	2.800	3.090
1.552	2.021	2.270	2.435	2.629	2.809	3.096
1.700	2.027	2.272	2.478	2.633	2.818	3.128
1.803	2.055	2.274	2.490	2.642	2.821	3.233
1.861	2.063	2.301	2.511	2.648	2.848	3.433
1.865	2.098	2.301	2.514	2.684	2.88	3.585
1.944	2.14	2.359	2.535	2.697	2.954	3.585
1.958	2.179	2.382	2.554	2.726	3.012	

TABLE 4: Tensile Strength data probability distribution estimates.

Distribution	Parameter Estimates		LL	AIC	AICc	BIC
Exponential	$\hat{\beta}$=2.451333		-130,8676	263,7352	263,7949	265,9693
Lindley	$\hat{\beta}$=1.527872		-119,1903	240,3805	240,4402	244,6146
Gamma	$\hat{\alpha}$=23.384274	$\hat{\beta}$=0,104828	-50,0374	104,0747	104,2565	108,5429
Weibull	**$\hat{\alpha}$=5.504851**	**$\hat{\beta}$=2,650859**	**-49,5961**	**103,1923**	**103,3741**	**107,6605**
Log-Normal	$\hat{\mu}$=0,875096	$\hat{\sigma}^2$=0,045109	-51,3841	106,7683	106,9501	111,2365
Log-Logistic	$\hat{\alpha}$=8,483858	$\hat{\beta}$=2,43079	-50,7744	105,5488	105,7307	110,0170
Pareto	$\hat{\alpha}$=1,656883	$\hat{\beta}$=1,312	-94,5409	193,0818	193,2636	197,5500

TABLE 5: Confidence Intervals for Tensile Strength data.

Parameter	Estimate	Srd. Error	Z-test	p-Value	%95 Confidence Intervals			
					Asymptotic Approach		LRT Approach	
β	1,527872	0,117561	12,996	<0.001	Lower L	Upper L	Lower L	Upper L
					1,297452	1,758292	1,289828	1,826279

Simulation Example-1: Uncensored Data

In this section, we investigate the behaviour of the ML estimators for a finite sample size ($n=50$). Simulation study based on ($\beta=5$) Lindley distribution is carried out. The random variables are generated by using inverse transformation method. The generated function is given as;

$$t_i : 0{,}6\mu_i - (6+t_i)e^{-0{,}2t_i} = 0 \tag{19}$$

It is found for t_i value given in Equation (19) has uniform distribution with $U \sim (0;1)$. Simulation Data-I is given in Table.6:

Parameter estimates, and model selection criteria results for all methods are given in Table.6. As for results given in Table.7, the Lindley distribution gave the best result according to the AICC and BIC criterions and the second-best result according to the LL and AIC criterions. In Figure.2, it is given confidence intervals for both hazard and survival functions with %95.

TABLE 6: Simulation data-I (uncensored) (n=50).

Waiting time (In minutes)														
0.37	0.69	0.96	1.51	1.78	2.00	2.53	2.81	3.14	3.34	3.42	3.62	3.2	3.3	3.5
3.68	3.72	4.16	4.24	4.27	5.10	5.62	5.90	5.95	6.22	6.23	6.57	4.8	4.9	4.9
6.61	6.73	7.13	7.13	7.73	7.91	8.32	8.39	9.41	9.84	9.94	10.44	7.1	7.1	7.1
10.48	10.76	10.78	10.78	11.07	11.70	12.46	12.48	15.15	15.29	15.63	16.64	8.9	9.5	9.6
21.65	22.91													

TABLE 7: Simulation data (uncensored) Probability distribution estimates.

Distribution	Parameter Estimates		LL	AIC	AICc	BIC
Exponential	$\hat{\beta}$=7.7132		-152.1471	306.2942	306.3775	308.2062
Lindley	$\hat{\beta}$=4.261571		-147.0207	296.0414	296.1247	297.9534
Gamma	$\hat{\alpha}$=1.970504	$\hat{\beta}$=3.914329	-146.5631	297.1262	297.3815	300.9502
Weibull	$\hat{\alpha}$=1.540034	$\hat{\beta}$=8.56179	-146.0106	296.0212	296.2765	299.8452
Log-Normal	$\hat{\mu}$=1.768242	$\hat{\sigma}^2$=0.845412	-150.9627	305.9253	306.1807	309.7493
Log-Logistic	$\hat{\alpha}$=1.953769	$\hat{\beta}$=7.078896	-148.2085	299.4111	301.2599	302.7673
Pareto	$\hat{\alpha}$=0.361991	$\hat{\beta}$=0.37	-189.2190	382.4381	382.6934	386.2621

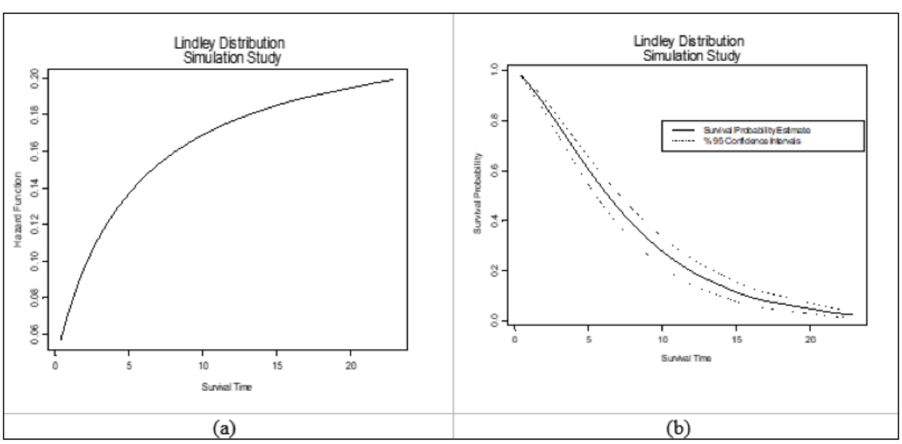

FIGURE 2: For Simulation Data: (a) hazard and (b) survival functions with %95 CI's.

Simulation Example-2: Censored Data

Second simulation example is right censored data and it is given in Table.8. In this example, Lindley parameter is calculated as $\hat{\beta}$=4,700359. Thus, the estimation of survival function is given with

$$\hat{S}(t)= \frac{(4{,}700359+t_i+1)}{(4{,}700359+1)} \exp\left(-\frac{t_i}{4{,}700359}\right).$$

Parameter estimates, and model selection criteria results for all methods are given in Table.9. As for results given in Table.9, the Lindley distribution gave the best result according to the AIC, AICC and BIC criterions.

TABLE 8: Simulation Data-I (+ censored).

Waiting time (In minutes)														
0.37	0.69	0.96	1.51	1.78	2.00	2.53	2.81	3.14	3.34	3.42	3.62	3.2	3.3	3.5
3.62	3.68	4.16	4.24	4.27	5.62	5.95	6.22	6.23	6.57	6.61	6.57	4.8	4.9	4.9
6.73	7.13	7.50	7.73	8.39	9.84	9.94	10.44	10.76	10.88	11.07	10.44	7.1	7.1	7.1
11.70	12.46	12.48	15.15	15.29	15.63	16.64	21.65	22.91	3.72+	5.10+	16.64	8.9	9.5	9.6
5.90+	7.91+	8.32+	9.41+	10.48+	10.78+									

TABLE 9: Simulation Data-II (censored) Probability distribution estimates.

Distribution	Parameter Estimates		LL	AIC	AICC	BIC
Exponential	$\hat{\beta}$=7.7132		-135.1260	272.2520	272.3353	274.8572
Lindley	$\hat{\beta}$=4.261571		-131.4311	264.8622	264.9455	267.4674
Gamma	$\hat{\alpha}$=1.970504	$\hat{\beta}$=3.914329	-131.4306	266.8612	267.1112	272.0715
Weibull	$\hat{\alpha}$=1.540034	$\hat{\beta}$=8.56179	-130.8538	265.7076	265.9576	270.9179
Log-Normal	$\hat{\mu}$=1.768242	$\hat{\sigma}^2$=0.845412	-135.1403	274.2806	274.5306	279.4979
Log-Logistic	$\hat{\alpha}$=1.953769	$\hat{\beta}$=7.078896	-134.2085	272.4170	272.6670	277.6273
Pareto	$\hat{\alpha}$=0.361991	$\hat{\beta}$=0.37	-164.5392	333.0784	333.3284	338.2887

Simulation Example-3: Uncensored Data

The simulation data used in Example-3 is derived for 200 observations. The simulation data is generated as follows:

Algorithm: Calculate the following values for i = 1,2,3,…,n

Step.1: Calculate n number of uniform random variables with ($\mu_i \sim U(0;1)$)

Step.2: Calculate n number of exponential random variables with mean of β ($x_i \sim E(\beta)$)

Step.3: Calculate n number of exponential random variables with parameters of α and β ($y_i \sim G(\alpha,\beta)$)

Step.4: $$t_i = \begin{cases} if\ \mu_i \le \dfrac{1}{\beta+1}\ ;\ choose\ x_i \\ if\ \mu_i > \dfrac{1}{\beta+1}\ ;\ choose\ y_i \end{cases}$$

Simulation Data-III for 200 samples is given in Table.10:

Parameter estimates, and model selection criteria results for all methods are given in Table.11. Results for given in Table.11, it can be clearly seen that the Lindley distribution gave the best result according all model selection criterions.

In Table.12, it is given confidence intervals for Lindley coefficient given in Table.11:

TABLE 10: Simulation Data-III (uncensored) (n=200).

Waiting time (In minutes)									
0.099	1.498	3.364	4.908	6.172	7.405	9.156	11.735	13.48	17.156
0.219	1.563	3.607	4.924	6.184	7.586	9.179	11.864	13.51	17.272
0.270	1.624	3.640	4.963	6.308	7.692	9.483	11.900	13.539	17.919
0.300	1.694	3.715	5.023	6.335	7.713	9.525	11.904	13.636	19.009
0.319	1.744	3.797	5.053	6.420	7.737	9.622	11.981	13.666	19.865
0.370	1.818	3.839	5.213	6.554	7.784	9.792	12.023	13.787	20.083
0.490	1.883	4.030	5.223	6.590	7.875	9.962	12.208	13.933	21.211
0.545	1.949	4.086	5.224	6.636	8.023	10.031	12.416	13.944	21.454
0.606	2.146	4.362	5.309	6.789	8.029	10.166	12.468	14.100	23.253
0.607	2.208	4.376	5.335	6.796	8.078	10.241	12.547	14.299	23.560
0.675	2.391	4.496	5.371	6.962	8.241	10.282	12.675	14.321	23.975
0.739	2.754	4.534	5.542	7.018	8.244	10.360	12.728	14.350	25.208
0.752	2.768	4.535	5.634	7.092	8.430	10.748	12.792	14.383	25.611
0.848	2.780	4.623	5.643	7.093	8.522	10.817	12.906	15.517	25.988
0.904	2.820	4.658	5.730	7.096	8.590	10.961	12.944	15.884	27.652
1.036	3.013	4.681	5.773	7.201	8.681	11.141	13.097	16.012	27.686
1.120	3.085	4.714	6.066	7.285	8.701	11.402	13.106	16.026	28.575
1.171	3.132	4.826	6.121	7.301	8.722	11.465	13.206	16.533	29.440
1.338	3.267	4.870	6.150	7.356	8.942	11.545	13.306	16.858	33.395
1.489	3.276	4.893	6.162	Tem.38	8.967	11.574	13.472	16.872	36.943

TABLE 11: Simulation Data-III (uncensored) Probability distribution estimates.

Distribution	Parameter Estimates		LL	AIC	AICc	BIC
Exponential	$\hat{\beta}$=8.977440		-638.9430	1279.8860	1279.906	1283.1840
Lindley	$\hat{\beta}$=4.904032		-629.3020	**1260.6040**	**1260.624**	**1265.9022**
Gamma	$\hat{\alpha}$=1.465608	$\hat{\beta}$=6.125404	-630.9614	1265.9228	1265.9837	1272.5194
Weibull	$\hat{\alpha}$=1.300242	$\hat{\beta}$=9.696552	-629.1979	1262.3958	1262.4567	1268.9924
Log-Normal	$\hat{\mu}$=1.81624	$\hat{\sigma}^2$=1.079451	-654.6809	1313.3617	1313.4226	1319.9584
Log-Logistic	$\hat{\alpha}$=1.831586	$\hat{\beta}$=6.964715	-646.3824	1296.7647	1296.8257	1303.3614
Pareto	$\hat{\alpha}$=0.242197	$\hat{\beta}$=0.099	-846.8489	1697.6978	1697.7587	1704.2945

TABLE 12: Confidence Intervals for Simulation Data-III.

Parameter	Estimate	Srd. Error	Z-test	p-Value	%95 Confidence Intervals			
					Asymptotic Approach		LRT Approach	
					Lower L	Upper L	Lower L	Upper L
β	4,904032	0.207162	23.672	<0.001	4.497995	5.310069	4.450043	5.421494

DISCUSSION

Censoring in a study is when there is incomplete information about a study participant, observation or value of a measurement. In clinical trials, it's when the event doesn't happen while the subject is being monitored or because they drop out of the trial. It is often difficult to work with censored data because of this particular structure. In many survival analysis methods, the distribution for censored data is passed from one eye to another. In this study, point and interval estimation for Lindley distribution with censored and uncensored data is introduced.

In application part, two real time (uncensored) and three simulations (uncensored and censored) numerical examples are tested. Estimates from Lindley distribution compared with results obtained from Exponential, Gamma, Weibull, Log-Normal, Log-Logistic and Pareto distributions. Five different information criteria were used to compare the results. As for the results; it can clearly have said that Lindley distribution has the best result for censored and uncensored simulations. Similarly, in the real time studies, it is seen that the best results belong to the Gamma distribution, and the second-best estimate belongs to the Lindley distribution.

In recent years, Lindley distribution has been used frequently in survival analysis. In many studies, the relationship between the distribution of Lindley and other distributions used in survival analysis was investigated and tested. Bhati et al. were studied Lindley Exponential distribution and the results were compared with New Generalized Lindley, Power Lindley, Lindley, Weibull and Exponential distributions.[12] In the comparisons made on 2 simulation data, it was found that the best result belongs to the Lindley Exponential distribution. Zakerzadeh and Dolati were studied Generalized Lindley Exponential distribution and the results were compared with Gamma, Weibull and Lognormal distributions.[18] They compared 2 real-time data with the distributions mentioned and stated that Generalized Lindley Exponential distribution has the best result. Cakmakyapan and Ozel were studied Lindley-Weibull and Lindley-Lomax distributions with 3 real-time data.[19] They compared results with Weibull, Lomax, Lindley, Exponential, Standard Lomax and Extended Lomax distributions. As for results, they stated that Lindley distributions with Weibull and Lomax have the best results.

It was stated that the results obtained from the studies carried out with the Lindley distribution gave more successful results especially in some data structures compared to other survival analysis methods. The data used in most of the studies are uncensored data structure. In this study, the effectiveness of Lindley distribution in censored and uncensored data structures was investigated and demonstrated.

CONCLUSION

In this study, the usage of the Lindley distribution for censored and uncensored data has been demonstrated in survival analysis. It can be said that the results obtained from mentioned data were successful according to similar distributions used in survival analysis. Precisely determining the distribution of data to be used in survival analysis is one of the prerequisites for the analysis to be more successful. The Lindley distribution, with its increasing popularity in recent years, is a distribution that gives successful results in survival analysis both censored and uncensored data.

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Conflict of Interest

No conflicts of interest between the authors and / or family members of the scientific and medical committee members or members of the potential conflicts of interest, counseling, expertise, working conditions, share holding and similar situations in any firm.

Authorship Contributions

Idea/Concept: *Kamil Alakuş;* ***Design:*** *Necati Alp Erilli;* ***Control/Supervision:*** *Kamil Alakuş;* ***Data Collection and/or Processing:*** *Kamil Alakuş;* ***Analysis and/or Interpretation:*** *Kamil Alakuş;* ***References and Fundings:*** *Necati Alp Erilli.*

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