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A New-Two Parameter Sujatha Distribution with Properties and Applications

Yeni İki Parametreli Sujatha Dağılımı, Özellikleri ve Uygulamaları

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ABSTRACT In this paper we propose a new two-parameter Sujatha distribution (NTPSD), which includes one parameter Akash distribution and Sujatha distribution as particular cases. The shapes of probability density function for varying values of parameters have been studied. Its statistical properties including coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have been discussed. The estimation of parameters has been discussed using the method of moments and the method maximum likelihood. Applications of the distribution have been discussed with two real lifetime data and the goodness of fit found quite satisfactory over one parameter exponential, Lindley and Sujatha distributions and two- parameter Sujatha distribution.

Keywords: Sujatha distribution; Akash distribution; moments; statistical properties; estimation of parameters; applications

ÖZET Bu makalede özel durumlarda bir parametreli Akash dağılımı ve Sujatha dağılımını içeren iki parametreli Sujatha dağılımı (NTPSD) önerilmiştir. Parametrelerin farklı değerleri için olasılık yoğunluk fonksiyonunun şekilleri araştırılmıştır. Değişim katsayısı, çarpıklık, basıklık, dağılım indeksi, ölüm hızı fonksiyonu, ortalama artık yaşam fonksiyonu, stokastik sıralama, ortalama sapmalar, Bonferroni ve Lorenz eğrileri ve stress-dayanıklılık güvenirliliğini içeren, dağılımın istatistiksel özellikleri ele alınmıştır. Moment ve en çok olabilirlik yöntemini kullanarak parametre tahminleri tartışılmıştır. İki gerçek yaşam verisi ile dağılımın uygulaması ele alınmış ve uyum iyiliği bir parametreli üstel, Lindley ve Sujatha dağılımı ve iki parametreli Sujatha dağılımına göre oldukça tatmin edici bulunmuştur.

Anahtar Kelimeler: Sujatha dağılımı; Akash dağılımı; momentler; parametre tahminleri; uygulamalar

Shanker (2016a) proposed a one parameter lifetime distribution named Sujatha defined by its probability density function (pdf) and cumulative distribution function (cdf)[1]

$$f_1(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x};\ x>0,\theta>0 \tag{1.1}$$

$$F_1\left(x;\theta\right)=1-\left[1+\frac{\theta x\left(\theta x+\theta+2\right)}{\theta^2+\theta+2}\right]e^{-\theta x};\ x>0,\theta>0 \tag{1.2}$$

where θ is a scale parameter. Shanker (2016a) has introduced this distribution as a convex combination of exponential θ, a gamma $(2,\theta)$ and a gamma $(3,\theta)$ distributions for modeling of lifetime data.[1] It has been found that Sujatha distribution gives better fit than exponential distribution and Lindley distribution proposed by Lindley (1958).[2]

The first four moments about origin and central moments of Sujatha distribution obtained by Shanker (2016a) are[1]

$$\mu_1' = \frac{\theta^2+2\theta+6}{\theta\left(\theta^2+\theta+2\right)}, \qquad \mu_2' = \frac{2\left(\theta^2+3\theta+12\right)}{\theta^2\left(\theta^2+\theta+2\right)},$$

$$\mu_3' = \frac{6\left(\theta^2+4\theta+20\right)}{\theta^3\left(\theta^2+\theta+2\right)}, \qquad \mu_4' = \frac{24\left(\theta^2+5\theta+30\right)}{\theta^4\left(\theta^2+\theta+2\right)}$$

$$\mu_2 = \frac{\theta^4+4\theta^3+18\theta^2+12\theta+12}{\theta^2\left(\theta^2+\theta+2\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^6+6\theta^5+36\theta^4+44\theta^3+54\theta^2+36\theta+24\right)}{\theta^3\left(\theta^2+\theta+2\right)^3}$$

$$\mu_4 = \frac{3\left(3\theta^8+24\theta^7+172\theta^6+376\theta^5+736\theta^4+864\theta^3+912\theta^2+480\theta+240\right)}{\theta^4\left(\theta^2+\theta+2\right)^4}$$

Shanker (2016a) has discussed its important statistical properties including shapes of density function for varying values of parameters, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability.[1] Shanker (2016a) has discussed the maximum likelihood estimation of parameter and showed the applications of Sujatha distribution to model lifetime data from biomedical science and engineering.[1] Shanker (2016b) has introduced Poisson- Sujatha distribution (PSD), a Poisson mixture of Sujatha distribution, and studied its properties, estimation of parameter and applications to model count data.[3] Shanker and Hagos (2015) have discussed zero-truncated Poisson- Sujatha distribution (ZTPSD) and applications for modeling count data excluding zero counts.[4] Shanker and Hagos (2016) have also studied size-biased Poisson- Sujatha distribution and its applications for count data excluding zero counts.[5]

Shanker (2015) introduced Akash distribution having parameter θ and defined by its pdf[6]

$$f_2\left(x,\theta\right) = \frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x};\ x>0,\ \theta>0 \tag{1.3}$$

Akash distribution is a convex combination of exponential (θ) and a gamma $(3,\theta)$ distributions. Shanker (2015) has discussed its statistical properties including shapes of pdf shapes of pdf, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual function, stress-strength reliability, distribution of order statistics, mean deviations, stochastic ordering, and Boneferroni and Lorenz curves , some among others. The estimation of parameter using maximum likelihood has been discussed by Shanker (2015).[6] Shanker (2015) has also discussed the applications of Akash distribution for modeling lifetime data from engineering and biomedical science and established its superiority over exponential and Lindley distribution.[6]

Recently, Mussie and Shanker (2018) proposed a two-parameter Sujatha distribution (TPSD) defined by is pdf [7]

$$f_3(x;\theta,\alpha)=\frac{\theta^3}{\alpha\theta^2+\theta+2}\left(\alpha+x+x^2\right)e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \tag{1.4}$$

Mussie and Shanker (2018) have discussed various statistical properties, estimation of parameters and application of TPSD.

In this paper, a new two-parameter Sujatha distribution (NTPSD), which includes Akash distribution and Sujatha distribution as special cases, has been proposed. Itsmoments and moments based measures have been discussed. Statistical properties including skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, stress-strength reliability have been discussed. The estimation of the parameters has been discussed using methods of moments and method of maximum likelihood. Two numerical examples have been presented to test the goodness of fit of NTPSD over one parameter exponential, Lindley and Sujatha distributions and TPSD.

A NEW TWO-PARAMETER SUJATHA DISTRIBUTION

A new two-parameter Sujatha distribution (NTPSD) having parameters θ and α can be defined by its pdf.

$$f_4(x;\theta,\alpha)=\frac{\theta^3}{\theta^2+\alpha\theta+2}\left(1+\alpha x+x^2\right)e^{-\theta x};x>0,\theta>0,\alpha\geq 0 \tag{2.1}$$

where θ is a scale parameter and α is a shape parameter. It reduces to Sujatha distribution (1.1) and Akash distribution (1.3) for α= 1 and α= 0 respectively. Like Sujatha distribution (1.1) and TPSD (1.4), NTPSD (2.1) is also a convex combination of exponential (θ), gamma (2, θ) and gamma (3, θ) distributions as follow

$$f_4\left(x;\theta,\alpha\right)=p_1\,g_1\left(x;\theta\right)+p_2\,g_2\left(x;\theta,2\right)+\left(1-p_1-p_2\right)g_3\left(x;\theta,3\right) \tag{2.2}$$

where

$$p_1=\frac{\theta^2}{\theta^2+\alpha\theta+2},\ p_2=\frac{\alpha\theta}{\theta^2+\alpha\theta+2},\ g_1\left(x;\theta\right)=\theta e^{-\theta x};\ x>0,\ \theta>0$$

$$g_2\left(x;\theta,2\right)=\frac{\theta^2}{\Gamma(2)}e^{-\theta x}x^{2-1};x>0,\ \theta>0,\ g_3\left(x;\theta,3\right)=\frac{\theta^3}{\Gamma(3)}e^{-\theta x}x^{3-1};\ x>0,\ \theta>0.$$

The cdf of NTPSD (2.1) can be expressed as

$$F_2\left(x,\theta,\alpha\right)=1-\left[1+\frac{\theta x\left(\theta x+\alpha\theta+2\right)}{\theta^2+\alpha\theta+2}\right]e^{-\theta x};x>0,\theta>0,\alpha>0 \tag{2.3}$$

Behaviors of the pdf and the cdf of NTPSD for various combination of parameters θ and α are shown in figures 1 and 2 respectively.

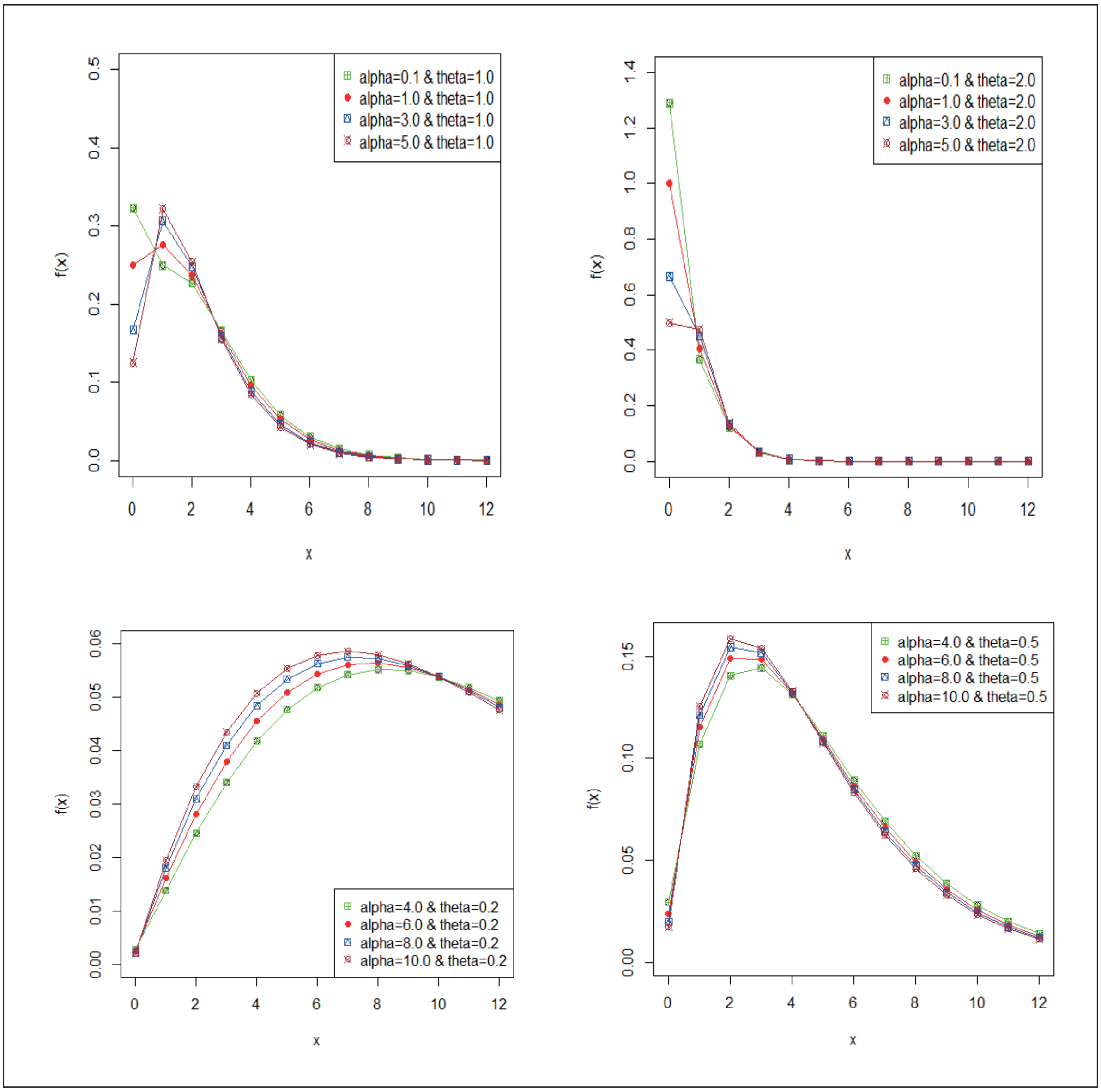

FIGURE 1: Behavior of the pdf of NTPSD for various combination of parameters θ and α.

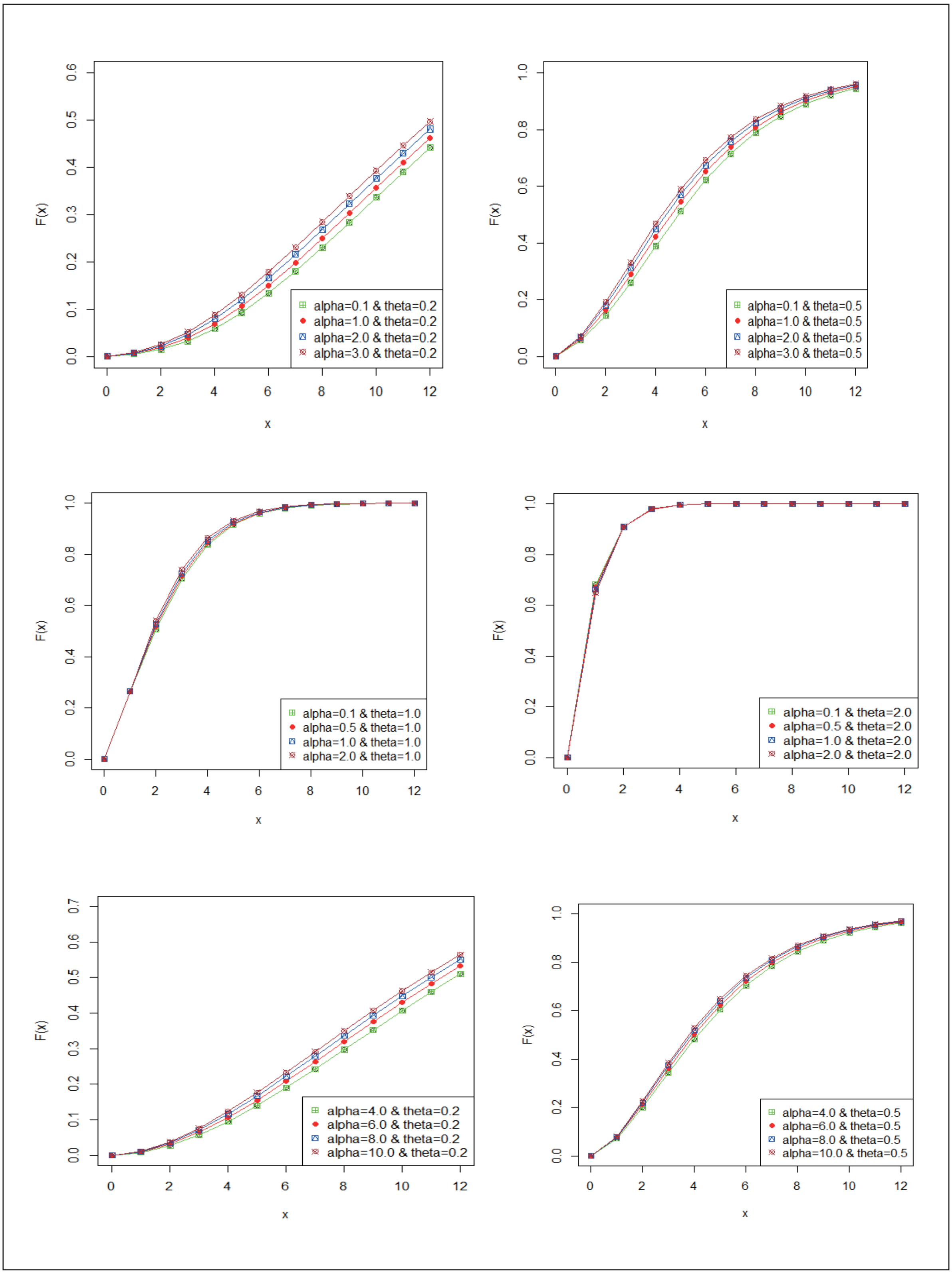

FIGURE 2: Behavior of the cdf of NTPSD for various combination of parameters θ and α

STATISTICAL CONSTANTS AND RELATED MEASURES

The r th moment about origin of NTPSD can be expressed as

$$\mu_r' = \frac{r!\left\{\theta^2 + (r+1)\alpha\theta + (r+1)(r+2)\right\}}{\theta^r\left(\theta^2 + \alpha\theta + 2\right)} \quad ; \; r = 1, 2, 3, \ldots \tag{3.1}$$

The first four moments about origin of NTPSD are obtained as

$$\mu_1' = \frac{\theta^2 + 2\alpha\theta + 6}{\theta\left(\theta^2 + \alpha\theta + 2\right)} \qquad \mu_2' = \frac{2(\theta^2 + 3\alpha\theta + 12)}{\theta^2\left(\theta^2 + \alpha\theta + 2\right)}$$

$$\mu_3' = \frac{6(\theta^2 + 4\alpha\theta + 20)}{\theta^3\left(\theta^2 + \alpha\theta + 2\right)} \qquad \mu_4' = \frac{24(\theta^2 + 5\alpha\theta + 30)}{\theta^4\left(\theta^2 + \alpha\theta + 2\right)}$$

The relationship between moments about the mean and moments about the origin gives the moments about mean of NTPSD as

$$\mu_2 = \frac{2\alpha^2\theta^2 + 4\alpha\theta^3 + \theta^4 + 16\theta^2 + 12\alpha\theta + 12}{\theta^2\left(\theta^2 + \alpha\theta + 2\right)^2}$$

$$\mu_3 = \frac{2\left(2\alpha^3\theta^3 + 6\alpha^2\theta^4 + 6\alpha\theta^5 + \theta^6 + 18\alpha^2\theta^2 + 42\alpha\theta^3 + 30\theta^4 + 36\alpha\theta + 36\theta^2 + 24\right)}{\theta^3\left(\theta^2 + \alpha\theta + 2\right)^3}$$

$$\mu_4 = \frac{3\begin{pmatrix} 8\alpha^4\theta^4 + 32\alpha^3\theta^5 + 44\alpha^2\theta^6 + 24\alpha\theta^7 + 3\theta^8 + 96\alpha^3\theta^3 + 320\alpha^2\theta^4 + 344\alpha\theta^5 \\ +128\theta^6 + 336\alpha^2\theta^2 + 768\alpha\theta^3 + 408\alpha\theta^4 + 480\alpha\theta + 576\theta^2 + 240 \end{pmatrix}}{\theta^4\left(\theta^2 + \alpha\theta + 2\right)^4}$$

The coefficients of variation (C.V), skewness $\left(\sqrt{\beta_1}\right)$, kurtosis $\left(\beta_2\right)$ and index of dispersion $\left(\gamma\right)$ of NTPSD are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{2\alpha^2\theta^2 + 4\alpha\theta^3 + \theta^4 + 16\theta^2 + 12\alpha\theta + 12}}{\theta^2 + 2\alpha\theta + 6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(2\alpha^3\theta^3 + 6\alpha^2\theta^4 + 6\alpha\theta^5 + \theta^6 + 18\alpha^2\theta^2 + 42\alpha\theta^3 + 30\theta^4 + 36\alpha\theta + 36\theta^2 + 24\right)}{\left(2\alpha^2\theta^2 + 4\alpha\theta^3 + \theta^4 + 16\theta^2 + 12\alpha\theta + 12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\begin{pmatrix} 8\alpha^4\theta^4 + 32\alpha^3\theta^5 + 44\alpha^2\theta^6 + 24\alpha\theta^7 + 3\theta^8 + 96\alpha^3\theta^3 + 320\alpha^2\theta^4 + 344\alpha\theta^5 + 128\theta^6 \\ +336\alpha^2\theta^2 + 768\alpha\theta^3 + 408\theta^4 + 480\alpha\theta + 576\theta^2 + 240 \end{pmatrix}}{\left(2\alpha^2\theta^2 + 4\alpha\theta^3 + \theta^4 + 12\alpha\theta + 16\theta^2 + 12\right)^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{2\alpha^2\theta^2+4\alpha\theta^3+\theta^4+16\theta^2+12\alpha\theta+12}{\theta\left(\theta^2+\alpha\theta+2\right)\left(\theta^2+2\alpha\theta+6\right)}$$

It can be easily verified that these statistical constants of NTPSD reduce to the corresponding statistical constants of Sujatha distribution and Akash at $\alpha=1$ and $\alpha=0$ respectively.

The behaviors of C.V., $\sqrt{\beta_1}$, β_2 and γ, for various combination of parameters θ and α have been shown numerically in tables 1, 2, 3, and 4

For a given value of α, C.V increases as the value of θ increases. Similarly for value $0\le\theta\le0.5$ C.V increases as the value of α increases .But for values $1\le\theta\le5$, C.V decreases as the value of α increases.

Since $\sqrt{\beta_1}>0$, NTPSD is always positively skewed, and this means that NTPSD is a suitable model for positively skewed lifetime data.

TABLE 1: C.V. of NTPSD for various combination of parameters θ **and** α.

α \ θ	0.2	0.5	1	2	3	4	5
0.2	0.592108	0.646393	0.767777	0.942081	1.007743	1.025809	1.028072
0.5	0.596998	0.653155	0.765466	0.920447	0.983663	1.005900	1.012842
1	0.604466	0.662392	0.761739	0.892143	0.95119	0.977525	0.989835
2	0.617257	0.675561	0.755148	0.853461	0.904948	0.934118	0.95194
3	0.627771	0.68427	0.749753	0.828221	0.873548	0.902632	0.922559
4	0.636530	0.690291	0.745356	0.810435	0.850803	0.878801	0.899315
5	0.643909	0.694600	0.741736	0.797217	0.833556	0.860148	0.880533

TABLE 2: Coefficient of skewness $\left(\sqrt{\beta_1}\right)$ of NTPSD for various combination of parameters θ and α.

α \ θ	0.2	0.5	1	2	3	4	5
0.2	1.135896	1.097674	1.186213	1.592857	1.872351	2.007995	2.064315
0.5	1.138983	1.116957	1.21277	1.567307	1.810817	1.938183	1.999736
1	1.145006	1.145839	1.247611	1.535588	1.733747	1.848046	1.912879
2	1.158918	1.192975	1.294584	1.496069	1.635857	1.728229	1.790774
3	1.173717	1.229048	1.324052	1.473222	1.577689	1.653369	1.710061
4	1.188372	1.257088	1.343787	1.458796	1.540048	1.603006	1.653482
5	1.202394	1.279255	1.357664	1.449096	1.514195	1.612083	1.612083

TABLE 3: Coefficient of kurtosis $\left(\beta_2\right)$ of NTPSD for various combination of parameters θ and α.

α \ θ	0.2	0.5	1	2	3	4	5
0.2	4.953644	4.81737	4.915277	6.342026	7.840435	8.778639	9.257092
0.5	4.96005	4.865924	5.022933	6.283795	7.545203	8.3676	8.832468
1	4.973635	4.944566	5.170213	6.21499	7.193906	7.868405	8.297711
2	5.008243	5.087074	5.380592	6.135306	6.779046	7.260076	7.61302
3	5.048363	5.207486	5.520332	6.092757	6.550496	6.912737	7.202118
4	5.090687	5.307836	5.617728	6.067446	6.410449	6.693844	6.933765
5	5.133284	5.391435	5.688264	6.051201	6.318161	6.546286	6.747854

TABLE 4: Index of dispersion (γ) of NTPSD for various combination of parameters θ and α.

α \ θ	0.2	0.5	1	2	3	4	5
0.2	5.157743	2.293584	1.363176	0.748843	0.472754	0.330238	0.249134
0.5	5.196202	2.303704	1.339286	0.72619	0.464444	0.328846	0.250377
1	5.252329	2.313480	1.305556	0.696429	0.452381	0.325758	0.251067
2	5.340332	2.317016	1.254545	0.655556	0.433551	0.318826	0.249815
3	5.403901	2.309910	1.217949	0.628788	0.419697	0.312319	0.247229
4	5.449819	2.298422	1.190476	0.60989	0.409142	0.306645	0.244351
5	5.482783	2.285380	1.169118	0.595833	0.400855	0.301783	0.241548

Since $\beta_2 > 3$, NTPSD is always leptokurtic, which means that NTPSD is more peaked than the normal curve. Thus NTPSD is suitable for lifetime data which are leptokurtic.

As long as $0 \le \theta \le 1$ and $0 \le \alpha \le 5$, the nature of NTPSD is over dispersed $\left(\sigma^2 > \mu_1'\right)$ and for $1 \le \theta \le 5$ and $0 \le \alpha \le 5$, the nature of TPSD is over dispersed $\left(\sigma^2 < \mu_1'\right)$.
The behavior of C.V., $\sqrt{\beta_1}$, β_2 and γ, for selected values of parameters θ and α are shown in figure3.

RELIABILITY PROPERTIES

HAZARD RATE FUNCTION AND MEAN RESIDUAL LIFE FUNCTION

Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$.The hazard rate function (also known as failure rate function), $h(x)$ and the mean residual function, $m(x)$ of X are respectively defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{p(X < x + \Delta x \mid X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)}$$

$$\text{and } m(x) = E[X - x \mid X > x] = \frac{1}{1 - F(x)} \int_x^\infty [1 - F(t)]\, dt$$

The corresponding, $h(x)$ and $m(x)$ of NTPSD (2.1) are thus obtained as

$$h(x) = \frac{\theta^3 \left(1 + \alpha x + x^2\right)}{\theta^2 \left(1 + \alpha x + x^2\right) + 2\theta x + \alpha\theta + 2}$$

$$\text{and} \quad m(x) = \frac{\theta^2 + \alpha\theta + 2}{\left[\left(\theta^2 + \alpha\theta + 2\right) + \theta x\left(\theta x + \alpha\theta + 2\right)\right] e^{-\theta x}} \int_x^\infty \left[1 + \frac{\theta t\left(\theta t + \alpha\theta + 2\right)}{\theta^2 + \alpha\theta + 2}\right] e^{-\theta t} dt$$

$$= \frac{\theta^2 x^2 + \theta(\alpha\theta + 4)x + \left(\theta^2 + 2\alpha\theta + 6\right)}{\theta\left[\theta^2 x^2 + \theta(\alpha\theta + 2)x + \left(\theta^2 + \alpha\theta + 2\right)\right]}$$

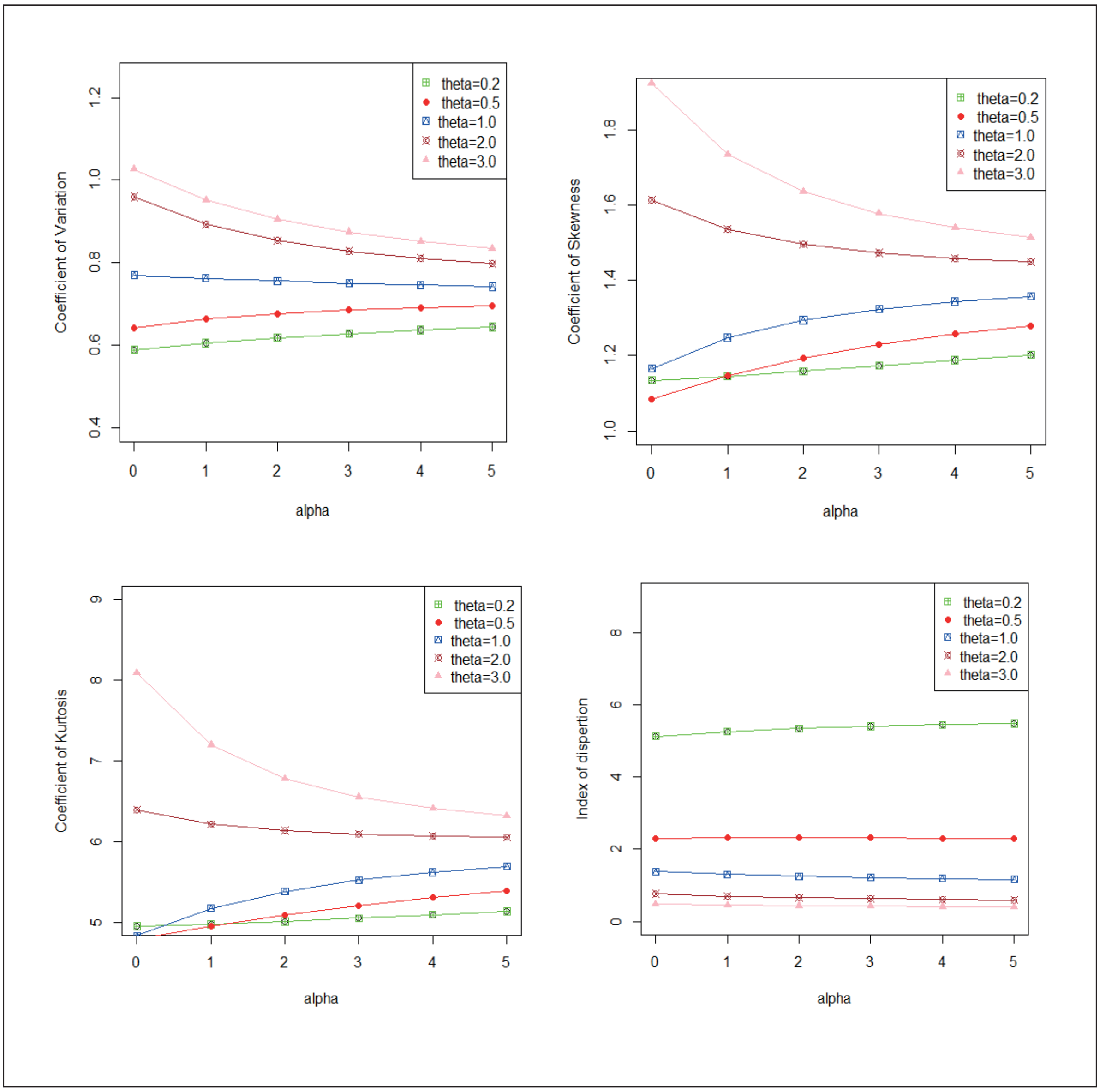

FIGURE 3: Behavior of C.V., $\sqrt{\beta_1}$, β_2 and γ for various combination of parameters θ and α.

It can be easily verified that $h(0)=\dfrac{\theta^3}{\theta^2+\alpha\theta+2}=f(0)$ and $m(0)=\dfrac{\theta^2+2\alpha\theta+6}{\theta\left(\theta^2+\alpha\theta+2\right)}=\mu_1'$.

The expression for $h(x)$ and $m(x)$ of NTPSD reduce to the corresponding $h(x)$ and $m(x)$ of Sujatha distribution at $\alpha=1$. Behaviors of $h(x)$ and $m(x)$ of NTPSD (2.1) for various combination of parameters are shown in figures 4 and 5 respectively.

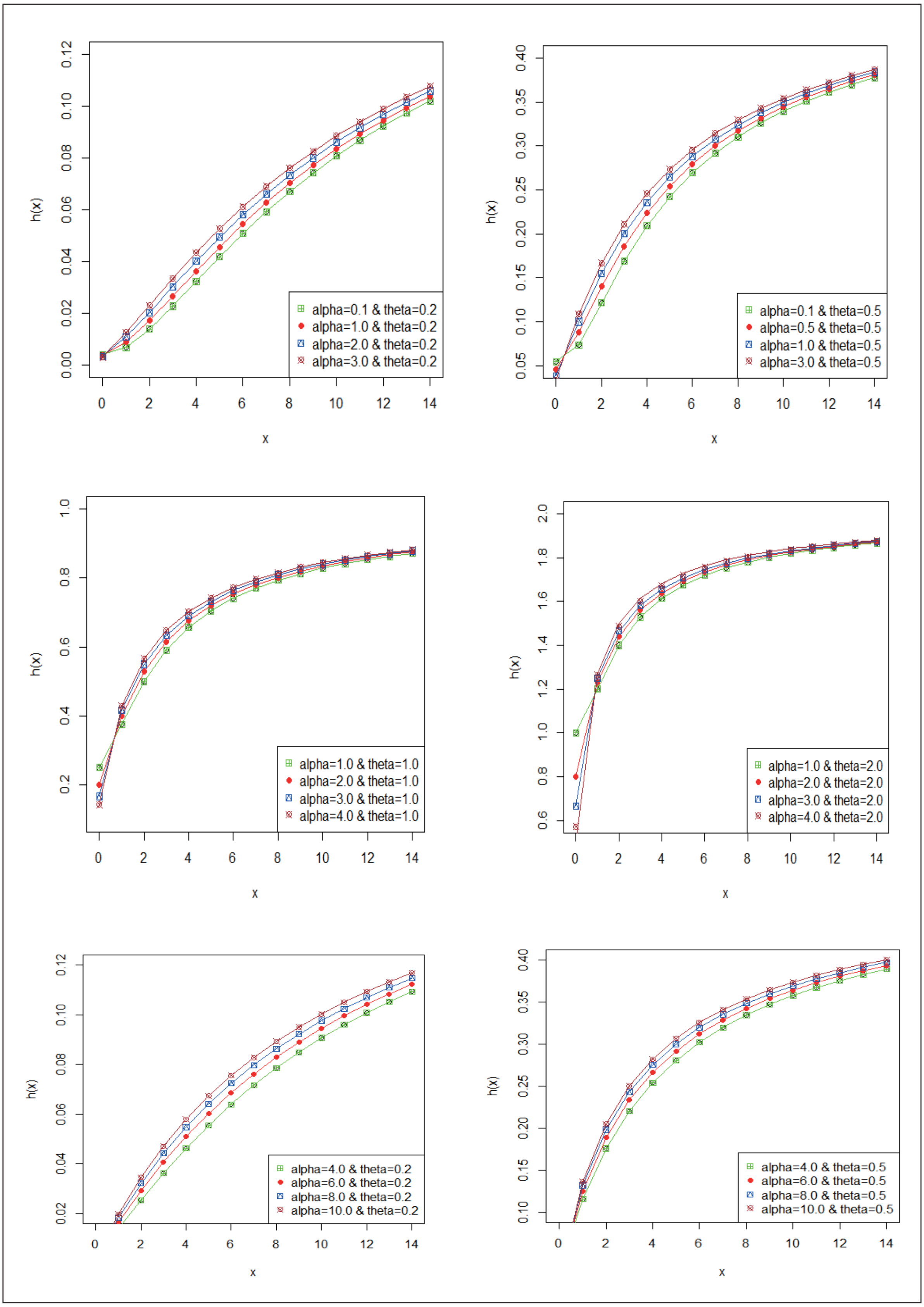

FIGURE 4: Behavior of $h(x)$ of NTPSD for various combinations of parameters θ and α.

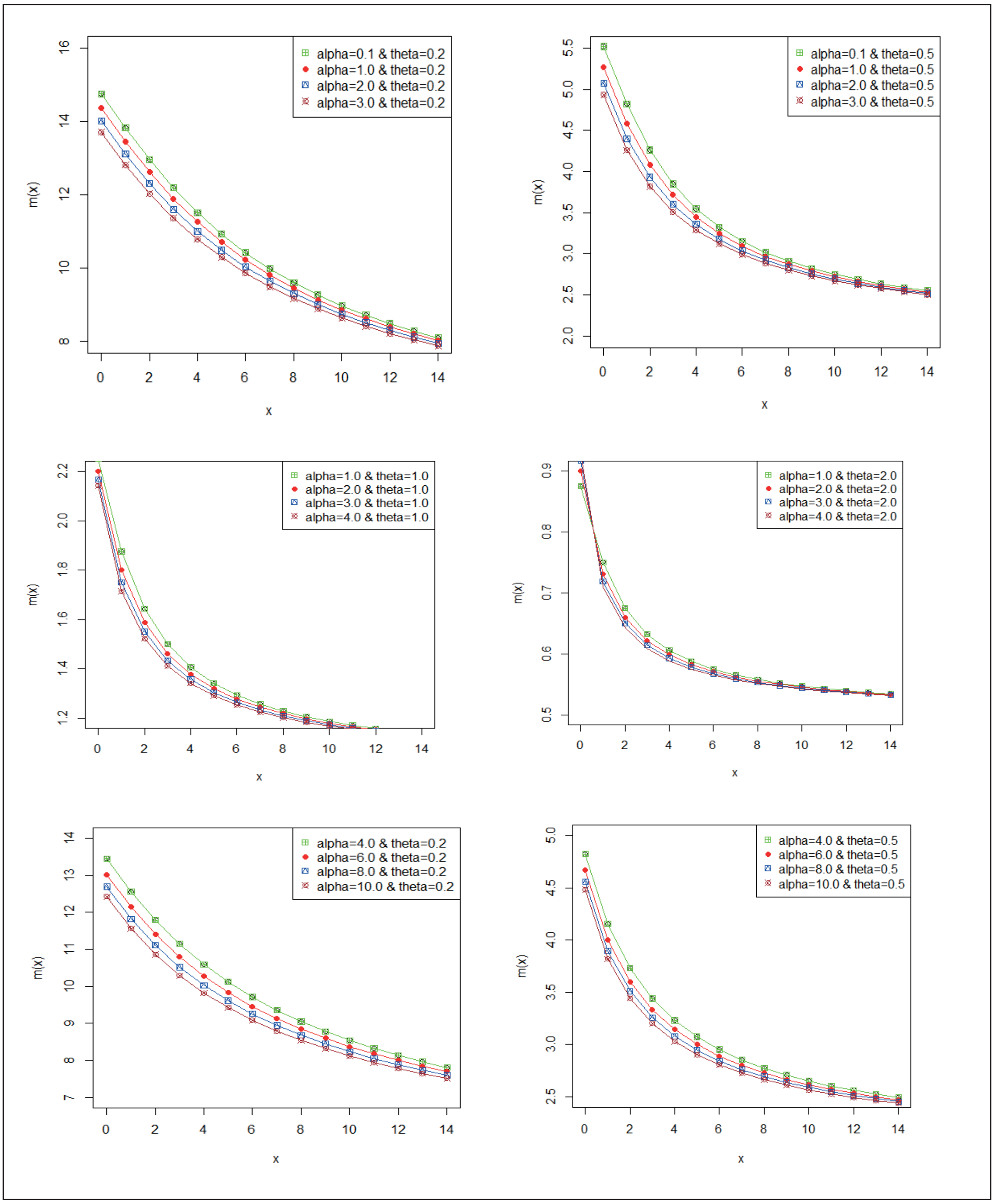

FIGURE 5: Behavior of $m(x)$ of NTPSD for various combinations of parameters θ and α.

It is obvious that $h(x)$ is monotonically increasing function of x, θ and α where as $m(x)$ is monotonically decreasing function of x, θ and α.

STOCHASTIC ORDERING

A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \leq_{st} Y\right)$ if $F_x(x) \geq F_y(x)$ for all x

(ii) hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_x(x) \geq h_y(x)$ for all x

(iii) mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_x(x) \leq m_y(x)$ for all x

(iv) likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\dfrac{f_x(x)}{f_y(x)}$ decreases in x

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions[8]

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mlr} Y$$
$$\Downarrow$$
$$x \leq_{st} y$$

The NTPSD (2.1) is ordered with respect to the strongest "likelihood ratio" ordering as expressed in the following theorem:

Theorem: Suppose $X \sim \text{NTPSD}(\theta_1, \alpha_1)$ and $Y \sim \text{NTPSD}(\theta_2, \alpha_2)$. If $\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$ (or $\theta_1 = \theta_2$ and $\alpha_1 \geq \alpha_2$) then $X \leq_{lr} Y$ and bence $X \leq_{hr} Y, X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof: We have

$$\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \frac{\theta_1^3\left(\theta_2^2+\alpha_2\theta_2+2\right)}{\theta_2^3\left(\theta_1^2+\alpha_1\theta_1+2\right)}\left(\frac{1+\alpha_1 x+x^2}{1+\alpha_2 x+x^2}\right)e^{-(\theta_1-\theta_2)x}; x>0$$

$$\ln\frac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \ln\left[\frac{\theta_1^3\left(\theta_2^2+\alpha_2\theta_2+2\right)}{\theta_2^3\left(\theta_1^2+\alpha_1\theta_1+2\right)}\right] + \ln\left(\frac{1+\alpha_1 x+x^2}{1+\alpha_2 x+x^2}\right) - \left(\theta_1-\theta_2\right)x$$

This gives $\dfrac{d}{dx}\ln\dfrac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} = \dfrac{(\alpha_1-\alpha_2)-(\alpha_1-\alpha_2)x^2}{\left(1+\alpha_1 x+x^2\right)\left(1+\alpha_2 x+x^2\right)} - \left(\theta_1-\theta_2\right).$

Thus, for $\left(\theta_1 > \theta_2 \text{ and } \alpha_1 = \alpha_2\right)$ or $\left(\alpha_1 > \alpha_2 \text{ and } \theta_1 = \theta_2\right)$ $\dfrac{d}{dx}\ln\dfrac{f_X(x;\theta_1,\alpha_1)}{f_Y(x;\theta_2,\alpha_2)} < 0$.

This means that $X \leq_{lr} Y$ and bence $X \leq_{hr} Y, X \leq_{mrl} Y$ and $X \leq_{st} Y$. This shows the flexibility of NT-PSD over Sujatha distribution and Akash distribution.

STRESS-STRENGTH RELIABILITY

Stress-strength reliability of a component shows the life of the component. When the stress of the component applied to it exceeds the strength X of the component, the component fails immediately and the component will function satisfactorily till $X > Y$. Thus, $R = P(Y < X)$ is a measure of the component reliability and is known as stress-strength reliability. It has applications including medical science, engineering, demography, economics, etc.

Suppose X and Y be independent strength and stress random variables having NTPSD (2.1) with parameters (θ_1, α_1) and (θ_2, α_2), respectively. Then, the stress-strength reliability R of NTPSD (2.1) can be obtained as

$$R = P(Y < X) = \int_0^\infty P(Y < X \mid X = x) f_x(x)\, dx$$

$$= \int_0^\infty f_3(x; \theta_{1,} \alpha_1) F_2(x; \theta_{2,} \alpha_2)\, dx$$

$$= 1 - \frac{\theta_1^3 \begin{bmatrix} \theta_2^6 + (\alpha_1 + \alpha_2 + 4\theta_1 + 1)\theta_2^5 + \left(\alpha_1\alpha_2 + 3\alpha_1\theta_1 + 4\alpha_2\theta_1 + 6\theta_1^2 + 2\alpha_1 + 3\theta_1 + 8\right)\theta_2^4 \\ + \left(3\alpha_1\alpha_2\theta_1 + 3\alpha_1\theta_1^2 + 6\alpha_2\theta_1^2 + 4\theta_1^3 + 4\alpha_1\theta_1 + 3\theta_1^2 + 18\alpha_1 + 2\alpha_2 + 22\theta_1 + 12\right)\theta_2^3 \\ + \left(3\alpha_1\alpha_2\theta_1^2 + \alpha_1\theta_1^3 + 4\alpha_2\theta_1^3 + \theta_1^4 + 2\alpha_1\theta_1^2 + \theta_1^3 + 26\alpha_1\theta_1 + 4\alpha_2\theta_1 + 22\theta_1^2 + 12\theta_1 + 52\right)\theta_2^2 \\ + \left(\alpha_1\alpha_2\theta_1^2 + \alpha_2\theta_1^3 + 10\alpha_1\theta_1 + 2\alpha_2\theta_1 + 10\theta_1^2 + 32\right)\theta_1\theta_2 + 2\left(\theta_1^2 + \alpha_1\theta_1 + 2\right)\theta_1^2 \end{bmatrix}}{\left(\theta_1^2 + \alpha_1\theta_1 + 2\right)\left(\theta_2^2 + \alpha_2\theta_2 + 2\right)(\theta_1 + \theta_2)^5}$$

It can be easily verified that the stress-strength reliability of Sujatha distribution is a special case of stress-strength reliability of NTPSD at $\alpha_1 = \alpha_2 = 1$.

STATISTICAL PROPERTIES

MEAN DEVIATIONS

The mean deviation about the mean and the mean deviation about the median are defined as

$\delta_1(X) = \int_0^\infty |x - \mu| f(x)\, dx$ and $\delta_2(X) = \int_0^\infty |x - M| f(x)\, dx$, respectively.

where $\mu = E(X)$ and $M = Median(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the following relationships

$$\delta_1(X) = 2\mu F(\mu) - 2\int_0^\mu x f(x)\, dx \tag{5.1.1}$$

and

$$\delta_2(X) = \mu - 2\int_0^M x f(x)\, dx \tag{5.1.2}$$

Using the pdf and expression for the mean of NTPSD, we get

$$\int_0^\mu x f_4(x;\theta,\alpha)\, dx = \mu - \frac{\left[\theta^3\left(\mu^3 + \alpha\mu^2 + \mu\right) + \theta^2\left(3\mu^2 + 2\alpha\mu + 1\right) + 2\theta(3\mu + \alpha) + 6\right] e^{-\theta\mu}}{\theta\left(\theta^2 + \alpha\theta + 2\right)} \tag{5.1.3}$$

$$\int_0^M x f_4(x;\theta,\alpha)\, dx = \mu - \frac{\left[\theta^3\left(M^3 + \alpha M^2 + M\right) + \theta^2\left(3M^2 + 2\alpha M + 1\right) + 2\theta(3M + \alpha) + 6\right] e^{-\theta M}}{\theta\left(\theta^2 + \alpha\theta + 2\right)} \tag{5.1.4}$$

Using expressions from (5.1.1), (5.1.2), (5.1.3) and (5.1.4) and after some tedious algebraic simplifications, $\delta_1(X)$ and $\delta_2(X)$ of NTPSD are expressed as

$$\delta_1(X)=\frac{2\left[\theta^2\left(\mu^2+\alpha\mu+1\right)+2\theta(2\mu+\alpha)+6\right]e^{-\theta\mu}}{\theta\left(\theta^2+\alpha\theta+2\right)} \tag{5.1.5}$$

$$\delta_2(X)=\frac{2\left[\theta^3\left(M^3+\alpha M^2+M\right)+\theta^2\left(3M^2+2\alpha M+1\right)+2\theta(3M+\alpha)+6\right]e^{-\theta M}}{\theta\left(\theta^2+\alpha\theta+2\right)}-\mu \tag{5.1.6}$$

BONFERRONI AND LORENZ CURVES AND INDICES

The Bonferroni and Lorenz curves and Bonferroni and Gini indices given by Bonferroni (1930) have applications almost every fields of knowledge including economics to study income and poverty, reliability, demography and medical science *(Bonferroni CE. Elementi di Statistica Generale. Firenze: Seeber; 1930)*. The Bonferroni and Lorenz curves can be expressed as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,f(x)\,dx=\frac{1}{p\mu}\left[\int_0^\infty xf(x)\,dx-\int_q^\infty xf(x)\,dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty xf(x)\,dx\right] \tag{5.2.1}$$

and $$L(p)=\frac{1}{\mu}\int_0^q x\,f(x)\,dx=\frac{1}{\mu}\left[\int_0^\infty xf(x)\,dx-\int_q^\infty xf(x)\,dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty xf(x)\,dx\right] \tag{5.2.2}$$

Also these can be expressed as

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{5.2.3}$$

and $$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{5.2.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are expressed as

$$B=1-\int_0^1 B(p)\,dp \tag{5.2.5}$$

and $$G=1-2\int_0^1 L(p)\,dp \tag{5.2.6}$$

respectively.

Using pdf of NTPSD (2.1), we get

$$\int_q^\infty x\,f_4(x;\theta,\alpha)\,dx=\frac{\left\{\theta^3\left(q^3+\alpha q^2+q\right)+\theta^2\left(3q^2+2\alpha q+1\right)+2\theta(3q+\alpha)+6\right\}e^{-\theta q}}{\theta\left(\theta^2+2\alpha\theta+6\right)} \tag{5.2.7}$$

Using equations (5.2.7), (5.2.1) and (5.2.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\theta^3\left(q^3+\alpha q^2+q\right)+\theta^2\left(3q^2+2\alpha q+1\right)+2\theta\left(3q+\alpha\right)+6\right\}\,e^{-\theta q}}{\theta^2+2\alpha\theta+6}\right] \tag{5.2.8}$$

and $L(p)=1-\dfrac{\left\{\theta^3\left(q^3+\alpha q^2+q\right)+\theta^2\left(3q^2+2\alpha q+1\right)+2\theta\left(3q+\alpha\right)+6\right\}\,e^{-\theta q}}{\theta^2+2\alpha\theta+6}$ (5.2.9)

Using equations (5.2.8) and (5.2.9) in (5.2.5) and (5.2.6), the Bonferroni and Gini indices of NTPSD (2.1) are obtained as

$$B=1-\frac{\left\{\theta^3\left(q^3+\alpha q^2+q\right)+\theta^2\left(3q^2+2\alpha q+1\right)+2\theta\left(3q+\alpha\right)+6\right\}e^{-\theta q}}{\theta^2+2\alpha\theta+6} \tag{5.2.10}$$

$$G=-1+\frac{2\left\{\theta^3\left(q^3+\alpha q^2+q\right)+\theta^2\left(3q^2+2q+1\right)+2\theta\left(3q+\alpha\right)+6\right\}e^{-\theta q}}{\theta^2+2\alpha\theta+6} \tag{5.2.11}$$

ESTIMATION OF PARAMETERS

In this section, the estimations of parameters of NTPSD using method of moments and method of maximum likelihood have been discussed.

METHOD OF MOMENT ESTIMATES (MOME)

Equating the population mean to the sample mean, we have

$$\bar{x}=\frac{\theta^2+2\alpha\theta+6}{\theta\left(\theta^2+\alpha\theta+2\right)}=\frac{\theta^2+\alpha\theta+2}{\theta\left(\theta^2+\alpha\theta+2\right)}+\frac{\alpha\theta+4}{\theta\left(\theta^2+\alpha\theta+2\right)}$$

$$\bar{x}=\frac{1}{\theta}+\frac{\alpha\theta+4}{\theta\left(\theta^2+\alpha\theta+2\right)}$$

$$\left(\theta^2+\alpha\theta+2\right)=\frac{\alpha\theta+4}{\theta\bar{x}-1} \tag{6.1.1}$$

Again equating the second population moment with the corresponding sample moment, we have

$$m_2'=\frac{2\left(\theta^2+3\alpha\theta+12\right)}{\theta^2\left(\theta^2+\alpha\theta+2\right)}=\frac{2\left(\theta^2+\alpha\theta+2\right)}{\theta^2\left(\theta^2+\alpha\theta+2\right)}+\frac{4\left(\alpha\theta+5\right)}{\theta^2\left(\theta^2+\alpha\theta+2\right)}$$

$$m_2'=\frac{2}{\theta^2}+\frac{4\left(\alpha\theta+5\right)}{\theta^2\left(\theta^2+\alpha\theta+2\right)}$$

$$\theta^2 + \alpha\theta + 2 = \frac{4(\alpha\theta + 5)}{m_2'\theta^2 - 2} \tag{6.1.2}$$

Equations (6.1.1) and (6.1.2) give the following cubic equation in θ as

$$\alpha m_2'\theta^3 + 4\left(m_2' - \alpha\bar{x}\right)\theta^2 - 2(10\bar{x} - \alpha)\theta + 12 = 0 \tag{6.1.3}$$

Solving equation (6.1.3) using any iterative method such as Newton-Raphson method, Regula-Falsi method or Bisection method, method of moment estimation (MOME) $\tilde{\theta}$ of θ can be obtained and substituting the value of $\tilde{\theta}$ in equation (6.1.1), MOME $\tilde{\alpha}$ of α can be obtained as

$$\tilde{\alpha} = \frac{-\bar{x}\left(\tilde{\theta}\right)^3 - 2(\bar{x} - \tilde{\theta})\tilde{\theta} + 6}{\tilde{\theta}(\tilde{\theta}\bar{x} - 2)} \tag{6.1.4}$$

MAXIMUM LIKELIHOOD ESTIMATES (MLE'S)

Suppose $(x_1, x_2, x_3, \ldots x_n)$ be random sample from NTPSD (2.1). The log likelihood function is thus obtained as

$$\ln L = n\left[3\ln\theta - \ln(\theta^2 + \alpha\theta + 2)\right] + \sum_{i=1}^{n}\ln\left(1 + \alpha x_i + x_i^2\right) - n\theta\bar{x}$$

The maximum likelihood estimate (MLE's) $(\hat{\theta}, \hat{\alpha})$ of (θ, α) are then the solutions of the following non-linear equations

$$\frac{\partial \ln L}{\partial \theta} = \frac{3n}{\theta} - \frac{n(2\theta + \alpha)}{\theta^2 + \alpha\theta + 2} - n\bar{x} = 0 \tag{6.2.1}$$

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{n\theta}{\theta^2 + \alpha\theta + 2} + \sum_{i=1}^{n}\frac{x_i}{1 + \alpha x_i + x_i^2} = 0 \tag{6.2.2}$$

These two log likelihood equations cannot be expressed in closed forms and hence seems difficult to solve directly. The (MLE's) $(\hat{\theta}, \hat{\alpha})$ of (θ, α) can be computed directly by solving the log likelihood equation using Newton-Raphson iteration available in R-software till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained. The initial values of parameters θ and α are the MOME $(\tilde{\theta}, \tilde{\alpha})$ of the parameters (θ, α).

DATA ANALYSIS

In this section the applications of NTPSD have been discussed using maximum likelihood estimates of parameters with two real lifetime datasets. The first dataset is the waiting time (in minutes) of customers before service in a bank and the second dataset is the relief time (in minutes) of patients receiving analgesic. The main purpose of considering these two datasets are that previously one parameter exponential, Lindley and Sujatha distributions and two-parameter Sujatha distribution (TPSD) have been used to model these datasets. Since exponential, Lindley, Sujatha and TPSD are related to NTPSD, it is expected that NTPSD will provide a better fit for these datasets over the considered distributions. Although the suitability of NTPSD has been tested with several lifetime datasets, but only two examples are provided.

Data set 1: This data set is used by Ghitany et al (2008) for fitting the Lindley (1958) distribution[2,9]

0.8,	0.8,	1.3,	1.5,	1.8,	1.9,	1.9,	2.1,	2.6,	2.7,	2.9,	3.1,	3.2,
3.3,	3.5,	3.6,	4.0,	4.1,	4.2,	4.2,	4.3,	4.3,	4.4,	4.4,	4.6,	4.7,
4.7,	4.8,	4.9,	4.9,	5.0,	5.3,	5.5,	5.7,	5.7,	6.1,	6.2,	6.2,	6.2,
6.3,	6.7,	6.9,	7.1,	7.1,	7.1,	7.1,	7.4,	7.6,	7.7,	8.0,	8.2,	8.6,
8.6,	8.6,	8.8,	8.8,	8.9,	8.9,	9.5,	9.6,	9.7,	9.8,	10.7,	10.9,	11.0,
11.0,	11.1,	11.2,	11.2,	11.5,	11.9,	12.4,	12.5,	12.9,	13.0,	13.1,	13.3,	13.6,
13.7,	13.9,	14.1,	15.4,	15.4,	17.3,	17.3,	18.1,	18.2,	18.4,	18.9,	19.0,	19.9,
20.6,	21.3,	21.4,	21.9,	23.0,	27.0,	31.6,	33.1,	38.5				

Data set 2: This data set is available in Gross and Clark (1975, P. 105).[10]

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7	4.1	1.8	1.5
1.2	1.4	3	1.7	2.3	1.6							

The values of $-2\ln L$,AIC (Akaike Information Criterion) and K-S Statistic (Kolmogorov-Smirnov Statistic) for the above data sets have been computed for the considered distributions. The formulae for computing AIC and K-S Statistics are
$AIC = -2\ln L + 2k$, $BIC = -2\ln L + k\ln n$ and $\text{K-S} = \sup_x \left|F_n(x) - F_0(x)\right|$, where k = the number of parameters, n = the sample size, and the $F_n(x)$ = empirical distribution function. The best distribution corresponds to lower values of $-2\ln L$, AIC, and K-S statistic. The MLE $(\hat{\theta},\hat{\alpha})$ along with their standard errors, S.E $(\hat{\theta},\hat{\alpha})$, $-2\ln L$, AIC, K-S Statistic and p-value of the fitted distributions are presented in the Table 5.

Since the $-2\ln L$, AIC and K-S are minimum for NTPSD, it is obvious that NTPSD is the best among the considered distributions.

TABLE 5 : MLE's, S.E $(\hat{\theta},\hat{\alpha})$, $-2\ln L$, AIC, and K-S Statistics and P-value of the fitted distributions of the given datasets.

Data set	Distribution	MLE's	S.E	$-2\ln L$	AIC	K-S	P-value
1	NTPSD	$\hat{\theta}$= 0.2242 $\hat{\alpha}$= 30.604	0.0285 47.8069	634.86	638.86	0.0465	0.9822
	TPSD	$\hat{\theta}$= 0.2794 $\hat{\alpha}$= 1.9647	0.0200 2.3691	639.29	643.29	0.0795	0.5525
	Sujatha	$\hat{\theta}$= 0.2846	0.0163	639.64	641.64	0.0884	0.4147
	Lindley	$\hat{\theta}$= 0.1866	0.0133	638.07	640.07	0.0677	0.7494
	Exponential	$\hat{\theta}$= 0.1012	0.0101	658.04	660.04	0.1730	0.0050
2	NTPSD	$\hat{\theta}$= 1.0590 $\hat{\alpha}$= 42.4700	0.1647 69.7644	52.74	56.74	0.3243	0.0298
	TPSD	$\hat{\theta}$= 0.5880 $\hat{\alpha}$= 103.4800	0.1404 171.61	66.50	70.50	0.4477	0.0007
	Sujatha	$\hat{\theta}$= 1.1367	0.1498	57.50	59.50	0.3590	0.0116
	Lindley	$\hat{\theta}$= 0.8161	0.1361	60.50	62.50	0.3911	0.0044
	Exponential	$\hat{\theta}$= 0.5263	0.1177	65.67	67.67	0.4395	0.0009

CONCLUDING REMARKS

A new two-parameter Sujatha distribution (NTPSD) has been introduced which includes Akash distribution and Sujatha distribution as particular cases. Raw moments and central moments of NTPSD have been derived. The behaviors of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of NTPSD have been studied with various combinations of parameters. Behaviors of hazard rate function and mean residual life function of NTPSD have been studied with varying values of the parameters. The stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability of NTPSD have also been discussed. The estimation of parameters have been discussed with two well-known methods namely method of moment and the method of maximum. Two examples of real lifetime datasets have been presented to show the applications and goodness of fit of NTPSD over one parameter exponential, Lindley and Sujatha distributions and TPSD. The values of -2InL, AIC, K-S and p-values of the distributions shows that NTPSD gives much better fit. Therefore, NTPSD can be thought of an essential two-parameter lifetime distribution for modeling real lifetime data.

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REFERENCES

1. Shanker R. Sujatha distribution and its applications. Statistics in Transition New Series 2016a;17(3):391-410.
2. Lindley DV. Fiducial distributions and Bayes' theorem. Journal of the Royal Statistical Society Series B 1958;20(1):102-7.
3. Shanker R. The discrete Poisson-Sujatha distribution. International Journal of Probability and Statistics 2016b;5(1):1-9.
4. Shanker R, Hagos F. Zero-truncated poisson-sujatha distribution with applications. Journal of Ethiopian Statistical Association 2015;24:55-63.
5. Shanker R, Hagos F. Size-biased Poisson-Sujatha distribution with applications. American Journal of Mathematics and Statistics 2016;6(4):145-54.
6. Shanker R. Akash distribution and its applications. International Journal of Probability and Statistics 2015;4(3):65-75.
7. Mussie T, Shanker R. A two-parameter Sujatha distribution. Biometrics & Biostatistics International Journal 2018;7(3):188-97.
8. Shaked M, Shanthikumar JG. Stochastic Orders and Their Applications. 1st ed. New York: Academic Press; 1994. p.545.
9. Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its application. Mathematics Computing and Simulation 2008;78(4):493-506.
10. Gross AJ, Clark VA. Survival Distributions: Reliability Applications in the Biometrical Sciences. 1st ed. New York: John Wiley; 1975. p.331.