On Several New Generalized Entropy Optimization Methods

Yeni Genelleştirilmiş Entropi Optimizasyon Yöntemleri

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ABSTRACT Objective: In this study we have suggested new Generalized Entropy Optimization Methods (GEOM) for solving Entropy Optimization Problems (EOP) consisting of optimizing a given entropy optimization measure subject to constraints generated by given moment vector functions. These problems acquire in different scientific fields as statistics, information theory, biostatistics especially in survival data analysis and etc. **Material and Methods:** Mentioned problems in the form of $\mathrm{GEOP2}, \mathrm{GEOP3}$ based on $\mathrm{GEOP1}$ have Generalized Entropy Optimization Distributions: $\mathrm{GEOD2}$ in the form of $\mathrm{Min_D\,MaxEnt}$, $\mathrm{Max_D\,MaxEnt}$; $\mathrm{GEOD3}$ in the form of $\mathrm{Min_H\,MinxEnt}, \mathrm{Max_H\,MinxEnt}$, where H is the Jaynes optimization measure, D is Kullback-Leibler optimization measure. It should be noted that formulation of $\mathrm{GEOP1}$ uses only one optimization measure (H or D), however each of formulations of $\mathrm{GEOP2}, \mathrm{GEOP3}$ uses two measures H, D together. **Results:** GEOP 1,2,3 are conditional optimization problems which can be solved by Lagrange multipliers method. It must be noted that calculating Lagrange multipliers can be fulfilled by starting from arbitrary initial point for Newton approximations of constructed auxiliary equation. **Conclusion:** There are situations, for example in survival data analysis, when both MaxEnt and MinxEnt distributions are accepted to given statistical data (or distribution) in the sense of same goodness of fit test. For this reason, developed our methods to obtain distributions are fundamental in statistical analysis. Analogous generalized problems can be also considered by the virtue of other measures different from H, D in dependency of requirements of experimental situation.

Key Words: Entropy; weights and measures

ÖZET Amaç: Bu çalışmada, tarafımızdan verilen entropi optimizasyon ölçümünü verilen moment vektör fonksiyonlar yardımıyla elde edilmiş koşullar altında optimize eden problemlerin çözümü için yeni Genelleştirilmiş Entropi Optimizasyon Metotları (GEOM) geliştirilmiştir. Söz konusu problemler istatistik, enformasyon teorisi, biyoistatistik özellikle de sağkalım veri analizi gibi birçok farklı alanda ortaya çıkmaktadır. **Gereç ve Yöntemler:** GEOP1'e dayalı GEOP2 ve GEOP3 problemleri sırasıyla GEOD2: $\mathrm{Min_D\,MaxEnt}, \mathrm{Max_D\,MaxEnt}$ şeklinde ve GEOD3: $\mathrm{Min_H\,MinxEnt}, \mathrm{Max_H\,MinxEnt}$ şeklinde Genelleştirilmiş Entropi Optimizasyon Dağılımları'na sahiptir. Burada, H Jaynes optimizasyon ölçümü, D ise Kullback-Leibler optimizasyon ölçümüdür. Vurgulamak gerekir ki, GEOP1'in formülasyonu H veya D optimizasyon ölçümünden sadece birinden yararlanırken, $\mathrm{GEOP2}, \mathrm{GEOP3}$'ün formülasyonlarının her biri H ve D ölçümlerini birlikte yararlanmakla ifade edilir. **Bulgular:** GEOP 1,2,3 koşullu optimizasyon problemleri olduğundan Lagrange çarpanları yöntemiyle çözülebilir. Vurgulamak gerekir ki, Lagrange çarpanlarının hesaplanması Newton yaklaşım yönteminin kurulmuş ilave denklem yardımıyla keyfi başlangıç nokta seçmekle gerçekleştirilebilir. **Sonuç:** Öyle durumlar mevcuttur ki örneğin sağkalım veri analizinde aynı bir veriye hem MaxEnt ve hem de MinxEnt dağılımları uyum sağladığından dolayı her iki dağılım önemlidir. Bu nedenle, istatistiksel veri analizinde tarafımızdan geliştirilen metotlar önem taşımaktadır. Deneysel ortama bağlı olarak, benzer genelleştirilmiş problemler H ve D'den farklı optimizasyon ölçümleri için de düşünülebilir.

Anahtar Kelimeler: Entropi; ağırlıklar ve ölçümler

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Several optimization principles are formulated and methods realizing these principles are suggested in.[1-6] Optimization principles can be applied to different statistical problems.[1,7]

MaxEnt method has been employed to approximate the size distribution of U.S. family income and MaxEnt distributions were compared with two conventional income distributions.[7]

A development of entropy optimization methods is given.[8] For this purpose, by considering entropy optimization measure on the given set of moment vector functions special functional according to each measure is defined. Furthermore, via the moment vector functions giving the least and greatest values to mentioned functionals, GEO distribution is obtained. Each of these distributions is closest (or furthest) to the given a priori distribution in the corresponding measure.[9] In the mentioned studies also applications of mentioned generalizations to statistical problems are represented. A generalization of Entropy Optimization Problems (GEOP) is formulated, proposed sufficient conditions for the existence solution and suggested a new method based on a priori evalutions and Newton's methods for calculation of Lagrange multipliers.[10]

In the mentioned papers the approach to obtain MinMaxEnt and MaxMinxEnt distributions can be formulated as a generalization of Entropy Optimization Methods. EOM have important applications, especially in statistics, physics, engineering, economy, survival data analysis[11-13], fuzzy logic, wind energy and so on.[14,15] Moreover, there are many studies about the application of mentioned methods in the literature when known statistical distributions do not conform to statistical data, but the entropy optimization distributions conform well.[14-18] Generalized Entropy Optimization Methods (GEOM) have proposed distributions in the form of the MinMaxEnt, the MaxMaxEnt how close to or far from statistical data (or distribution) in the sense of H or D measures.

Different aspects and methods of investigations of EOM are considered.[19-21] The investigation of these problems leads to GEOP are stated and studied.[12] And also, same applications are given.[22-23]

Our study consists of following sections. In Section 2, Entropy Optimization Problem (EOP) and Generalized Entropy Optimization Problems (GEOP1,2,3) are introduced. In Section 3, the existence of solutions of GEOP 2,3 is given by Theorems 1, 2. In Section 4, the main results obtained from the study are summarized and problems for future applications are expressed.

FORMULATION OF EOP, GEOP1,2,3 AND GEOD1,2,3

Entropy Optimization Problem (EOP) and Generalized Entropy Optimization Problem (GEOP1) can be formulated in the following forms.[24-25]

EOP: Let $f^{(o)}(x)$ be given probability distribution of random variable X, L be an entropy optimization measure and $g(x)$ be a given moment vector function generating m moment constraints. It is required to obtain the distribution $f(x)$ corresponding to $g(x)$ giving extremum value to L.

GEOP1: Let $f^{(o)}(x)$ be given probability distribution of random variable X, L be an entropy optimization measure and K be a set of given moment vector functions. It is required to choose moment vector functions $g^{(1)}, g^{(2)} \in K$ such that $g^{(1)}(x)$ generates distribution $f^{(1)}(x)$ closest to $f^{(o)}(x)$, $g^{(2)}(x)$ generates distribution $f^{(2)}(x)$ furthest from $f^{(0)}(x)$ with respect to entropy optimization measure L. If L is taken as Shannon entropy measure H, then $f^{(1)}(x)$ is called $\mathrm{Min_H}$ MaxEnt distribution and $f^{(2)}(x)$ is called $\mathrm{Max_H}$ MaxEnt distribution. If L is taken as Kullback-Leibler measure D, then $f^{(1)}(x)$ is called $\mathrm{Min_D}$ MinxEnt distribution and $f^{(2)}(x)$ is called $\mathrm{Max_D}$ MinxEnt distribution.[25]

In this study, we give the following generalizations of EOP.

GEOP2: Let $f^{(0)}(x)$ be given probability distribution of random variable X. H be Jaynes entropy measure, D be Kullback-Leibler measure and K be a set of given moment vector functions $g(x)$ generating moment vector conditions. It is required to choose moment vector functions $g^{(1)}, g^{(2)} \in K$ such that $g^{(1)}(x)$ generates MaxEnt distribution $f^{(1)}(x)$ Kullback-Leibler measure D of which has minimum value, $g^{(2)}(x)$ generates MaxEnt distribution $f^{(2)}(x)$ Kullback-Leibler measure D of which has maximum value on K.

MaxEnt distribution generated by $g^{(1)} \in K$ we call as $\mathrm{Min_D}$ MaxEnt distribution, MaxEnt distribution generated by $g^{(2)} \in K$ we call $\mathrm{Max_D}$ MaxEnt distribution.

The $\mathrm{Min_D}$ MaxEnt distribution represents the MaxEnt distribution $f^{(1)}(x)$ which closest to $f^{(o)}(x)$, $\mathrm{Max_D}$ MaxEnt distribution represents the MaxEnt distribution $f^{(2)}(x)$ which furthest from $f^{(0)}(x)$ in the sense of D measure.

GEOP3: Let $f^{(0)}(x)$ be given probability distribution of random variable X. H be Jaynes entropy measure, D be Kullback-Leibler measure and K be a set of given moment vector functions $g(x)$ generating moment vector conditions. It is required to choose moment vector functions $g^{(1)}, g^{(2)} \in K$ such that $g^{(1)}(x)$ generates MinxEnt distribution $f^{(1)}(x)$ Jaynes measure H of which has minimum value, $g^{(2)}(x)$ generates MinxEnt distribution $f^{(2)}(x)$ Jaynes measure H of which has maximum value on K. MinxEnt distribution generated by $g^{(1)}(x)$ we call $\mathrm{Min_H}$ MinxEnt distribution, MinxEnt distribution generated by $g^{(2)}(x)$ we call $\mathrm{Max_H}$ MinxEnt distribution. It should be noted that $\mathrm{Min_H}$ MinxEnt distribution represents the MinxEnt distribution $f^{(1)}(x)$ which closest to $f^{(0)}(x)$, $\mathrm{Max_H}$ MinxEnt represents MinxEnt distribution $f^{(2)}(x)$ which furhest from $f^{(0)}(x)$ in the sense of H measure.

So, we have defined the following GEOD ̍s as solutions of problems GEOP1, GEOP2, GEOP3.

($\mathrm{Min_H}$ MaxEnt, $\mathrm{Max_H}$ MaxEnt) is the solution of $GEOP1$ according to H measure,

($\mathrm{Min_D}$ MinxEnt, $\mathrm{Max_D}$ MinxEnt) is the solution of $GEOP1$ according to D measure,

($\mathrm{Min_D}$ MaxEnt, $\mathrm{Max_D}$ MaxEnt) is the solution of $GEOP2$ according to D measure,

($\mathrm{Min_H}$ MinxEnt, $\mathrm{Max_H}$ MinxEnt) is the solution of $GEOP3$ according to H measure.

The method of solving GEOP we call as Generalized Entropy Optimization Method (GEOM) and the solution of GEOP as Generalized Entropy Optimization Distribution (GEOD). From formulations of EOP and GEOP, it is shown that EOP deals with optimization of measure (H or D) for fixed moment vector function g, $g \in K$, however GEOP deals with optimization of measures H, D consecutively for all vector functions g, $g \in K$. In other words, the solution of GEOP is obtained by two stages:

i. Optimization of measure H or D for each fixed moment vector function $g \in K$,

ii. Optimization of measure H or D for all moment vector functions $g \in K$.

Moreover, formulation of GEOP1 is realized by using only one optimization measure (H or D), however statement of GEOP2,3 are represented via two entropy optimization measures (H, D) together.

THE EXISTENCE OF SOLUTIONS OF GEOP2, 3

GEOP2,3 can be solved simply for discrete random variables and continuous random variables when a set of moment vector functions K consists of finite number elements.

We give the following theorem for discrete random variables when a set of moment vector functions K consisting of finite number elements.

Theorem 1. Let H be Jaynes entropy measure,

$$H = -\sum_{i=1}^{n} p_i ln p_i \quad ; \tag{1}$$

$D(p;q)$ be Kullback-Leibler measure,

$$D(p;q) = \sum_{i=1}^{n} p_i \, ln \frac{p_i}{q_i} \quad ; \tag{2}$$

$p^{(0)} = (p_1^0, \ldots, p_n^0)$ be given statistical distribution; $K_0 = \{g_1, g_2, \ldots, g_r\}$ be given set of continuous linear independent characterizing $g(x)$ moment functions and all combinations of r elements of K_0 taken l elements at a time be $K_l (l = 1,2, \ldots, m; m+1 < n)$. Then, GEOP2 has a unique solution consisting of two distributions: Min_D MaxEnt, Max_D MaxEnt which are obtained by vector functions $g^{(1)}, g^{(2)} \in K_{1m}$, $K_{1m} = K_1 \cup K_2 \cup \ldots \cup K_m$.

Proof. Each element of K_l is vector function with l components and K_l has $r_l = \binom{r}{l}$ number of elements. The number of elements of K_{1m} is equal to $r_1 + r_2 + \cdots + r_m$, $(m+1 < n)$. In order to obtain GEOP2, it is required to consider entropy optimization measure H subject to constraints generated by vector function $(g_0(x), g(x))$, where $g_0 \equiv 1$, $g \in K_l$ $(l = 1,2, \ldots, m)$. Mentioned constraints can be expressed in the form

$$\sum_{i=1}^{n} p_i g_j(x_i) = \mu_j \ , \ (j = 0,1,2, \ldots, l) \tag{3}$$

where $\mu_0 = 1, g_0(x) = 1,\ \sum_{i=1}^{n} p_i^0 g_j(x_i) = \mu_j$, $j = 0,1,2, \ldots, l$.

Each vector function $(g_0(x), g(x))$, $g \in K_l$ generates l moment conditions and the number of distributions generated by $(g_0(x), g(x))$, $g \in K_l$ vector functions is equal to $r_l = \binom{r}{l}$. Then, the problem of solving GEOP2 consists of finding $r_1 + r_2 + \cdots + r_m$, $(m+1 < n)$ number of MaxEnt distributions and choosing among of these distributions the distribution for which optimization measure D reaches the least value and the distribution for which optimization measure D reaches the greatest value.

So, Min_D MaxEnt distribution is MaxEnt distribution, Kullback-Leibler measure of which is least, Max_D MaxEnt distribution is MaxEnt distribution, Kullback-Leibler measure of which is greatest among of all MaxEnt distributions.

It must be noted that each characterizing moment vector function $g \in K_{1m}$ together with $g_0 \equiv 1$ generates corresponding MaxEnt distribution. Consequently, Min_D MaxEnt, Max_D MaxEnt distributions are obtained by vector functions $g^{(1)}, g^{(2)} \in K_{1m}$, respectively.

Theorem 1 is proved.

The existence of Min_H MinxEnt and Max_H MinxEntdistributions as the solution of GEOP3 can be proved by similar method.

The existence of solutions GEOP2,3 for continous random variables can be established when a set of moment vector functions K is compact.

In the case of continuous random variables the existense of solution of GEOP2 can be formulated in the form of the following theorem.

Theorem 2. Let H be a Jaynes entropy measure

$$H = -\int_a^b f(x) \ln f(x)\, dx \quad ; \tag{1_1}$$

$D(f(x); q(x))$ be Kullback-Leibler measure,

$$D(f(x); q(x)) = -\int_a^b f(x) \ln \frac{f(x)}{q(x)} dx \quad ; \tag{2_1}$$

$f^{(0)}(x)$ be given statistical distribution; Kbe given set of continuous characterizing m dimensionel moment vector functions $g(x)$, $g = (g_1, g_2, \ldots, g_m)$.

If K be compact set of characterizing m dimensionel moment vector functions $g(x)$, then GEOP2 has a solution consisting of two distributions Min_D MaxEnt, Max_D MaxEnt which are obtained by vector functions $g^{(1)}, g^{(2)} \in K$ respectively.

Proof. Each MaxEnt distribution is obtained by vector function $(g_0(x), g(x))$, where $g_0 \equiv 1$, $\mu_0 = 1$, $g(x) = (g_1, g_2, \dots, g_m)$ generating moment conditions

$\int_a^b f(x) g_j(x) dx = \mu_j$, $j = 0,1,\dots,m$ (3_1)

where $\int_a^b f^{(0)}(x) g_j(x) dx = \mu_j$, $j = 0,1,\dots,m$,

and giving maximum value to functional (1_1). If functions $g_1(x), g_2(x), \dots, g_m(x)$ are linearly independent, then problem (1_1), (3_1) has a solution. Under this assumption variance-covariance matrix of random variables $g(x_1), g(x_2), \dots, g(x_m)$ are positive defined [33;34]. Consequently, each vector function $g \in K$, $g = (g_1, g_2, \dots, g_m)$ generates MaxEnt distribution giving maximum value to functional (1_1). If obtain MinxEnt measure of mentioned distribution, then $D(f(x): f^{(0)}(x))$ as continuous functional defined on compact set K reaches its minimum and maximum values at vector functions $g^{(1)}, g^{(2)} \in K$ respectively. Consequently, $g^{(1)} \in K$ which gives to $D$the minimum value generates $\mathrm{Min_D}$ MaxEnt distribution, $g^{(2)} \in K$ which gives to $D$the maximum value generates $\mathrm{Max_D}$ MaxEnt distribution.

Theorem 2 is proved.

By virtue of similar method the existence of distributions: $\mathrm{Min_H}$ MinxEnt, $\mathrm{Max_H}$ MinxEnt as the solution of problem GEOP3 can be proved.

GEOP 1,2,3 are conditional optimization problems which can be solved by Lagrange multipliers method . It must be noted that calculating Lagrange multipliers can be fulfilled by starting from arbitrary initial point for Newton's approximations of constructed auxiliary equation [12].

CONCLUSION

Generalized Entropy Optimization Methods modelling the statistical data in the form of Generalized Entropy Optimization Distributions can be successfully applied in many scientific fields. These distributions are closest the statistical data or furthest from this in the sense of measure H or D . Consequently, mentioned distributions can be used as a measure of closeness or farness in estimation of required distributions. Furthermore, it is known that in the dependence of the number and the type of characterizing moment vector functions it is possible all well and good approximate given statistical data. It should be noted that all GEOD's can be used in modelling, because by increasing the number and changing the type of characterizing moment vector functions MaxMaxEnt and MaxMinxEnt distributions also can be suitable for estimation.

Generalized Entropy Optimization Distributions generally are conditional optimization problems, consequently can be solved by Lagrange multipliers method. The solving mentioned problems leads to calculate Lagrange multipliers. The calculation of Lagrange multipliers can be fulfilled by starting from arbitrary initial point for Newton's approximations of constructed auxiliary equation. Results of applications of developed methods in this study will be given in other papers.

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